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Invariant Measures for Actions of Higher Rank Abelian Groups

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ABSTRACT. The first part of the paper begins with an introduction into Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ and an overview of the method of studying invariant measures for such actions based on consideration of conditional measures along various invariant foliations. The main body of that part contains a detailed proof of a modified version of the main theorem from [**KS3**] for actions by toral automorphisms of with applications to rigidity of the measurable structure of such actions with respect to Lebesgue measure. In the second part principal technical tools for studying nonuniformly hyperbolic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ are introduced and developed. These include Lyapunov characteristic exponents, nonstationary normal forms and Lyapunov Hoelder structures. At the end new rigidity results for $\mathbb{Z}^2$ actions on three-dimensional manifolds are outlined.

In this paper we discuss various results concerning invariant measures for actions of higher–rank abelian groups, i.e. $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geq 2$ on compact differentiable manifolds which display certain hyperbolic behavior. Similarly to the rank one case, hyperbolicity can be full or partial, and uniform or nonuniform. Full (corr. partial) uniform hyperbolicity appears for Anosov (corr. partially hyperbolic) actions. Nonuniform hyperbolicity appears for actions preserving measures for which all (for the full case) or some (for partial case) Lyapunov characteristic exponents do not vanish. Two parts of the paper deal with the uniform and nonuniform cases correspondingly.

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Part I. UNIFORMLY HYPERBOLIC ACTIONS

1. Definitions and examples

1.1. Anosov and partially hyperbolic actions.

1.1.1. *Anosov actions of discrete and continuous groups.* A smooth action α of a discrete group Γ on a compact manifold M is called *Anosov* if for some $\gamma \in \Gamma$, $\alpha(\gamma)$ is an Anosov diffeomorphism, *i.e.* the tangent bundle TM splits in an $\alpha(\gamma)$ invariant way into two subbundles (distributions) E^- and E^+ called the *stable* (or *contracting*) and *unstable* (or *expanding*) bundles (or distributions) correspondingly such that for some λ, $0 < \lambda < 1$ and $C \geq 1$ and for all positive integers m,

$$\|D(\alpha(\gamma^m))(v)\| \leq C\lambda^m, \tag{1.1.1}$$

if $v \in E^-$ and

$$\|D(\alpha(\gamma^m))(v)\| \geq C^{-1}\lambda^{-m}, \tag{1.1.2}$$

if $v \in E^+$.

Here $\|\cdot\|$ is the norm in TM generated by a Riemannian metric on M and D is the derivative map. The constant C (but not λ) depends on the choice of Riemannian metric.

More generally, a smooth locally free action of a Lie group G is called Anosov (or *normally hyperbolic*) if for a certain element conditions (1.1.1) and (1.1.2) hold for E^- and E^+ in the splitting $TM = E_o \oplus E^- \oplus E^+$, where E_o is the tangent bundle to the orbit foliation of the action. See [**KS1**] for a more detailed discussion. Such an element γ is then called *normally hyperbolic*. In this case the distributions E^- and E^+ are often called *the strong stable* and *the strong unstable* distributions.

The general theory of Anosov actions even for higher–rank abelian groups is not well developed. The first serious obstruction is the following open question.

Problem. *For an Anosov action of $\mathbb{R}^k$, $k \geq 2$ is the set $\mathcal{N}$ of normally hyperbolic elements dense?*

A similar question for $\mathbb{Z}^k$ actions is reduced to this one via suspension construction (Section 1.2.2). The set $\mathcal{N}$ is open and its connected components are convex cones. In all known examples $\mathbb{R}^k \setminus \mathcal{N}$ is the union of finitely many hyperplanes (*cf.* Lyapunov hyperplanes, Sections 1.2.3, 1.3, 5.2).

1.1.2. *Partially hyperbolic actions.* A more general class is formed by partially hyperbolic actions. A G action α is called *partially hyperbolic* if there is an $\gamma \in G$ and a $\alpha(\gamma)$–invariant splitting $TM = E^- \oplus E^+ \oplus E^0$, with E^- and E^+ satisfying (1.1.1) and (1.1.2) as before and E^0 the "slow" distribution with

$$(C')^{-1}\mu^{-m} \geq \|D(\alpha(\gamma^m))(v)\| \leq C'\mu^m \tag{1.1.3}$$

for some μ, $\lambda < \mu \leq 1$ and $C' \geq 1$.

1.1.3. *Contrast between rank one and higher rank.* Any discussion of Anosov actions of higher rank abelian groups requires a certain condition of being "genuinely higher rank" in order to avoid products of rank one actions and other situations which can be reduced to the rank one case. The structure of such actions in the higher–rank case looks quite different and much more rigid than in the classical rank one situation of diffeomorphisms ($\mathbb{Z}$–actions) and flows ($\mathbb{R}$-actions). All known examples of such actions satisfying the "genuine higher rank" assumptions are algebraic up to a differentiable conjugacy. In particular, most algebraic actions are known to be differentiably rigid [**KS2**]. For the algebraic actions a natural

condition is absence of rank one algebraic factors. (See e.g. Section 3.1, condition $(\mathcal{R}')$). Thus studying invariant measures for irreducible algebraic Anosov (and in certain cases, also partially hyperbolic) actions is quite natural.

One of the main differences between the rank one and higher rank situations may be highlighted as follows. In the former case the robust orbit structure is well–described by the Markov model via a Markov partition (see, e.g. [**KH**], Section 18.7). In particular, any invariant measure for the topological Markov chain associated with a Markov partition produces an invariant measure for the hyperbolic system in question. The variety of invariant measures, both well–structured and not, for a topological Markov chain, is huge; hence the same is true for the hyperbolic systems. In the higher–rank case the Markov model is not applicable, since natural topological Markov chains consist of maps with infinite topological entropy. Hence algebraic models should be understood more directly.

1.2. Higher Rank Actions by Toral Automorphisms. The most basic examples of smooth hyperbolic actions of higher rank abelian groups are actions by automorphisms of a torus and their suspensions.

1.2.1. *Preliminaries.* Denote by $GL(m,\mathbb{Z})$ the group of integral $m\times m$ matrices with determinant 1 or -1. Any matrix $A\in GL(m,\mathbb{Z})$ defines an automorphism of the torus $\mathbb{T}^m=\mathbb{R}^m/\mathbb{Z}^m$ which we denote by F_A.

The automorphism F_A is ergodic with respect to Lebesgue measure on $\mathbb{T}^m$ if and only if no eigenvalue of A is a root of unity.

This can be seen by considering the dual automorphism on the group of characters $\mathbb{Z}^m$. If A (and hence the transposed matrix A^t) has an r th root of unity as an eigenvalue there is a rational (and hence also an integer) invariant vector for $(A^t)^r$. Hence the operator $U_{F_A^r}$ induced by $(F_A)^r$ on L^2, has a nonconstant invariant function, the corresponding character χ, and $\sum_{i=0}^{r-1}\chi\circ(F_A^i)$ is an invariant nonconstant function for U_{F_A}.

On the other hand, if there are no roots of unity among the eigenvalues, all orbits of A^t acting on the characters, save the constants, are infinite and hence the operator $U_{F_A^r}$ has countable Lebesgue spectrum in the orthogonal complement to the constant and thus F_A is ergodic.

Furthermore, in the latter case A has an eigenvalue of absolute value greater then one (this can be seen by using a bit of linear algebra and a compactness argument), and F_A is a Bernoulli automorphism with respect to Lebesgue measure [**Kat**].

Any $\mathbb{Z}^k$ action α by automorphisms of $\mathbb{T}^m$ is given by an embedding $\rho_\alpha:\mathbb{Z}^k\to GL(m,\mathbb{Z})$ such that $\alpha(n)=F_{\rho_\alpha(n)}$, where $n=(n_1,...,n_k)\in\mathbb{Z}^k$.

Let α_1 and α_2 be two $\mathbb{Z}^k$ actions by automorphisms of $\mathbb{T}^{m_1}$ and $\mathbb{T}^{m_2}$ correspondingly. The action α_2 is called an *algebraic factor* of a_1 if there exists an epimorphism $h\colon\mathbb{T}^{m_1}\to\mathbb{T}^{m_2}$ such that $h\circ\alpha_1=\alpha_2\circ h$.

The action α is called *irreducible* if any algebraic factor has finite fibers, *i.e.* acts on the torus of the same dimension. Equivalently, α does not have any invariant subtorus of lower dimension, or ρ_α contains a matrix with irreducible characteristic polynomial [**B**]. In [**KS3, KS4**] irreducible actions are called completely irreducible.

1.2.2. *Suspension construction.* It will be more convenient for us to operate with $\mathbb{R}^k$ actions so we would like to pass from an action of $\mathbb{Z}^k$ to the corresponding action of $\mathbb{R}^k$. This is the so-called suspension construction.

Suppose $\mathbb{Z}^k$ acts on $\mathbb{T}^m$. Embed $\mathbb{Z}^k$ as a lattice in $\mathbb{R}^k$. Given a $\mathbb{Z}^k$ action α on the torus let $\mathbb{Z}^k$ act on $\mathbb{R}^k \times \mathbb{T}^m$ by

$$\bar{\alpha}(n)(t,x) = (t-n, \alpha(n)(x)) \tag{1.2.1}$$

and form the quotient

$$M = \mathbb{R}^k \times \mathbb{T}^m / \mathbb{Z}^k. \tag{1.2.2}$$

Note that the "vertical" action V of $\mathbb{R}^k$ on $\mathbb{R}^k \times \mathbb{T}^m$ by $V_s(t,x) = (t+s,x)$ commutes with the $\mathbb{Z}^k$-action and therefore descends to M. This $\mathbb{R}^k$-action is called the *suspension* of the $\mathbb{Z}^k$-action.

Note that any $\mathbb{Z}^k$-invariant measure on $\mathbb{T}^m$ lifts to a unique $\mathbb{R}^k$-invariant measure on M.

The manifold M is a fibration over the "time" torus $\mathbb{T}^k$ with the fiber $\mathbb{T}^m$. We note that TM splits into the direct sum $TM = T_f M \oplus T_o M$ where $T_f M$ is the subbundle tangent to the $\mathbb{T}^m$ fibers and $T_o M$ is the subbundle tangent to the orbit foliation.

Remark. The suspension construction is of very general nature. Namely, let α be an action of $\mathbb{Z}^k$ on a space X with a certain structure (measure, topology, differentiable, homogeneous, etc). Then (1.2.1) defines an action on the product and the factor (1.2.2) possesses the structure of the skew product over $\mathbb{T}^k$ with the fiber X and inherits the structure from the fiber. This structure is preserved by the suspension action of $\mathbb{R}^k$.

1.2.3. *Eigenvalues and Lyapunov Exponents.* Let us denote by $A_1, ..., A_k$ the generators of the action α, i.e. the images under ρ_α of the standard generators of $\mathbb{Z}^k$. For each A_i $\mathbb{R}^m$ splits as the direct sum of the root spaces of A_i:

$$\mathbb{R}^m = \bigoplus_{\lambda \in Sp A_i} Ker(A_i - \lambda)^m.$$

Since the matrices A_i commute there exists an invariant splitting $\mathbb{R}^m = \bigoplus V_j$ which is a common refinement of the above splittings. It is called the root decomposition for the action α. It follows that the tangent bundle $T\mathbb{T}^m = \mathbb{T}^m \times \mathbb{R}^m$ splits into the direct sum of the invariant subbundles corresponding to subspaces V_j. The Lyapunov exponent $\lambda(\alpha(n), v)$ exists for every element $\alpha(n) \in G_\alpha$ and every vector $v \in T\mathbb{T}^m$. If v lies in one of the subbundles then $\lambda(\alpha(n), v)$ equals to the logarithm of the absolute value of the corresponding eigenvalue of element $\alpha(n)$. Moreover, $\lambda(\alpha(\cdot), v)$ is an additive functional on $\mathbb{Z}^k$. Combining the subspaces V_j corresponding to the same Lyapunov exponent we obtain a more robust decomposition than the root decomposition which is called *the Lyapunov decomposition* for the action α. For the detailed discussion of Lyapunov exponents for $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions by automorphisms of tori and solenoids including the non-Archimedean exponents appearing from p-adic valuations, see [**KS3**, Section 2 and Appendix]. We will return to the discussion of Lyapunov exponents in greater generality, first for algebraic actions in Section 1.3, and then in the general setting in Section 5.

The Lyapunov exponent for the suspension corresponding to $T_o M$ is always identically zero. To exclude this trivial case, when we speak of Lyapunov exponents we will always mean the Lyapunov exponents corresponding to $T_f M$. These Lyapunov exponents of the $\mathbb{R}^k$ action are the extensions of the Lyapunov exponents of the $\mathbb{Z}^k$ action to the linear functionals on $\mathbb{R}^k$. The kernels of non-zero Lyapunov

exponents are called *Lyapunov hyperplanes.* Lyapunov exponents may be proportional to each other with positive or negative coefficients. In this case, they define the same Lyapunov hyperplane. The action is Anosov (Section 1.1) if there are no non-trivial identically zero Lyapunov exponents.

Definition. The action α is called *totally nonsymplectic (TNS)* if no Lyapunov exponents are proportional with a negative coefficient.

For a generalization of this concept see Section 5.2.

An element $a \in \mathbb{R}^k$ is called *regular* if it does not belong to any Lyapunov hyperplane. All other elements are called *singular.* We call a singular element *generic* if it belongs to only one Lyapunov hyperplane. A regular element for an Anosov action is called an *Anosov element.*

In the situation we are considering now, *i.e* for actions by automorphisms of the torus and for the suspensions of such actions, every Lyapunov subspace is uniquely integrable to a homogeneous foliation. This is not always the case for more general actions including homogeneous ones (Section 1.3); the *coarse Lyapunov decomposition* which combines positively proportional Lyapunov exponents is always integrable (Section 5.2).

For an element $a \in \mathbb{R}^k$ let us define the *stable, unstable* and *center* distributions E^-_a, E^+_a and E^0_a as the sum of the Lyapunov spaces for which the value of the corresponding Lyapunov exponent on a is negative, positive, and 0 respectively. For any singular element a the center distributions E^0_a always contains an invariant subdistribution E^I_a, called *isometric* distribution. For an element a we will denote the integral foliations of the stable, unstable, center and isometric distributions E^-_a, E^+_a, E^0_a and E^I_a by W^-_a, W^+_a, W^0_a and W^I_a correspondingly.

1.3. General algebraic actions. Now we will consider "essentially algebraic" partially hyperbolic (in particular Anosov) actions of either $\mathbb{R}^k$ or $\mathbb{Z}^k$ more general than actions by toral automorphisms and their suspensions.

We recall that an action of a group G on a compact manifold is *Anosov* if some element $g \in G$ acts normally hyperbolically with respect to the orbit foliation (Section 1.1; see [**KS1**] for more details).

1.3.1. *Affine actions of discrete groups.* To clarify the notion of an algebraic action, let us first define affine algebraic actions of discrete groups. Let H be a connected Lie group with $\Lambda \subset H$ a cocompact lattice. Define $\mathrm{Aff}(H)$ as the set of diffeomorphisms of H which map right invariant vector-fields on H to right invariant vectorfields. Define $\mathrm{Aff}(H/\Lambda)$ to be the diffeomorphisms of H/Λ which lift to elements of $\mathrm{Aff}(H)$. Finally, define an action ρ of a discrete group G on H/Λ to be *affine algebraic* if $\rho(g)$ is given by some homomorphism $G \to \mathrm{Aff}(H/\Lambda)$. Let $\mathfrak{h}$ be the Lie algebra of H. Identifying $\mathfrak{h}$ with the right invariant vectorfields on H, any affine algebraic action determines a homomorphism $\sigma : G \to \mathrm{Aut}\ \mathfrak{h}$. Call σ the *linear part* of this action. We will also allow quotient actions of these on finite quotients of H/Λ, e.g. on infranilmanifolds. For any Anosov algebraic action of a discrete group G, H has to be nilpotent (cf. eg. [**GS**], Proposition 3.13).

1.3.2. *Algebraic $\mathbb{R}^k$-actions.* Suppose $\mathbb{R}^k \subset H$ is a subgroup of a connected Lie group H. Let $\mathbb{R}^k$ act on a compact quotient H/Λ by left translations where Λ is a lattice in H. Suppose C is a compact subgroup of H which commutes with $\mathbb{R}^k$. Then the $\mathbb{R}^k$-action on H/Λ descends to an action on $M = C \setminus H/\Lambda$. The general algebraic $\mathbb{R}^k$-action ρ is a finite factor of such an action. Let $\mathfrak{c}$ be the Lie algebra of

C. The *linear part* of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c} \setminus \mathfrak{h}$ induced by the adjoint representation of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H.

Let us note that the suspension (Section 1.2.2) of an algebraic $\mathbb{Z}^k$-action is an algebraic $\mathbb{R}^k$-action (cf. [**KS1**], Section 2.2).

Let ρ be an algebraic action of $\mathbb{R}^k$, not necessarily Anosov. For a given $a \in \mathbb{R}^k$ we denote the strong stable foliation of $\rho(a)$ by $\mathcal{W}_a^-$, the strong stable distribution by E_a^-, and the 0-Lyapunov space by E_a^0. Note that E_a^0 is always integrable for algebraic actions. Denote the corresponding foliation by $\mathcal{W}_a^0$. For a ρ–invariant Borel probability measure μ let us denote by ξ_a the partition into ergodic components of the element $\rho(a)$, by $h_\mu(a)$ the measure–theoretic entropy of the map $\rho(a)$ and by $\pi(a)$ the π–partition of $\rho(a)$. Recall that $\pi(a)$ is in fact equal to the measurable hull $\xi(\mathcal{W}_a^-)$ of the partition into the leaves of the strong stable foliation $\mathcal{W}_a^-$.

1.3.3. *Lyapunov exponents.* Let us describe the structure of the linear part of an algebraic $\mathbb{R}^k$ actions in more detail. In fact, we will first consider the case on the actions on H/Λ. The Lie algebra $\mathfrak{h}$ splits into root spaces $\mathfrak{h}^\alpha, \alpha \in \Sigma$ of the adjoint representation. The roots are linear functionals on $\mathbb{R}^k$ and their real parts in fact coincide with the Lyapunov exponents of the action ρ. The positive half–space $\Re\alpha \geq 0$ is called a *Lyapunov half–space* and its boundary a *Lyapunov hyperplane.* Let us denote the sum of root spaces with a given Lyapunov half–space L by $\mathfrak{h}^L$. Combining roots with the same real parts and taking corresponding right–invariant distribution we obtain *Lyapunov distributions*; Lyapunov distributions for different Lyapunov exponents form the *Lyapunov decomposition.* Combining Lyapunov distributions with the same Lyapunov half–space one obtains *coarse Lyapunov distributions* which form the *coarse Lyapunov decomposition.*

The strong stable distribution E_a^- for an element $a \in \mathbb{R}^k$ equals the sum of Lyapunov distributions E^α for those α for which $\alpha(a) < 0$. In particular, E_a^- is the sum of certain coarse Lyapunov distributions. Conversely, any coarse Lyapunov distribution equals to the intersection of strong stable distributions for certain elements of the action. Thus, any coarse Lyapunov distribution integrates to a homogeneous foliation whose leaves are in fact cosets of a nilpotent subgroup of H. For the Lyapunov half–space L let us denote this nilpotent subgroup by N_L. By picking an element a on the boundary of L which does not lie on any other Lyapunov hyperplane (such an element is called *generic singular*) and an element b inside L we see that $E^L = E_a^0 \cap E_b^-$ and hence E_b^- is the direct sum of $E_{a_i}^0 \cap E_b^-$ for several generic singular elements α_i.

Notice that unlike the case of actions by automorphisms of a torus Lyapunov distributions may not be integrable.

The case of an action on a double coset space is not too different. In fact the kernel of the projection lies in the 0-Lyapunov distributions for *all* elements $a \in \mathbb{R}^k$. Thus the picture on M is essentially the same as in H/Λ, only the zero Lyapunov exponent has lower multiplicity. This is why, in particular, the double coset construction appears in many standard examples of Anosov $\mathbb{R}^k$ actions.

1.4. Conditional measures. We will study the rigidity of invariant measures based on understanding of their conditional measures on some natural invariant foliations. Let us briefly recall how a probability measure ν on a manifold M determines a system of conditional measures on a foliation F. For a more detailed overview see [**KS3**], Section 4.

Denote by $\mathcal{B}$ the Borel σ-algebra on M. A *measurable partition* ξ of M is a partition of M such that, up to a set of measure 0, the quotient space M/ξ is separated by a countable number of measurable sets [**R**]. For every x in a set of full ν-measure there is a probability measure ν_x^ξ defined on $\xi(x)$, the element of ξ containing x, and satisfying the following properties: If $\mathcal{B}_\xi$ is the sub-σ-algebra of $\mathcal{B}$ whose elements are unions of elements of ξ, and $A \subset M$ is a measurable set, then $x \mapsto \nu_x^\xi(A)$ is $\mathcal{B}_\xi$-measurable and $\nu(A) = \int \nu_x^\xi(A)\nu(dx)$. These conditions determine the measures ν_x^ξ uniquely.

Let F be a foliation with smooth leaves and let us denote by $F(x)$ the leaf through x. Even though all foliations that we are going to consider in the next section will be linear, the partition into the leaves of F is not a measurable partition in general. Conditional measures on the leaves of the foliations that we will be working with are σ-finite locally finite measures ν_x^F defined up to a multiplicative constant. In other words, for almost every $x \in M$ and for open sets $A, B \subset F(x)$ with compact closures one can canonically define *the ratio* $\frac{\nu_x^F(A)}{\nu_x^F(B)}$. The conditional measures can be defined as follows.

Let us call a measurable partition ξ *subordinate* to F if for ν-a.e. x we have $\xi(x) \subset F(x)$ and $\xi(x)$ contains a neighborhood of x open in the submanifold topology of $F(x)$. Note that two different partitions subordinate to the same foliation determine conditional measures that are scalar multiples when restricted to the intersection of an element of one partition with an element of the other partition. Thus there is a locally finite measure ν_x^F on $F(x)$ uniquely defined up to scaling that restricts to a scalar multiple of a conditional measure for each partition subordinate to F. The measures ν_x^F form the system of conditional measures on the leaves of F.

Let $\sigma(F)$ denote the σ-algebra of all sets that consist a.e. of complete leaves of F. It corresponds to a unique measurable partition which is called the *measurable hull* of F, and is denoted by $\xi(F)$. It is the finest measurable partition whose elements consist a.e. of the entire leaves of F.

Notice that if the conditional measures on leaves of a foliation are δ-measures as will often be the case in the course of our considerations then *a posteriori* the partition into leaves of the foliation turns out to be measurable, indeed equivalent to the partition ϵ of the space into single points.

The next proposition is one of the expressions of fact that entropy measure uncertainly in the future given complete knowledge of the past. In particular, zero entropy implies that the past uniquely determines the future. This proposition is a particular case of [**KS3**, Proposition 4.1].

PROPOSITION 1.1. *Let f be a diffeomorphisms of a compact manifold, W be a an f-invariant expanding foliation, i.e. $\|Df^1\|_{TW}\| < 1$, and μ be an f-invariant Borel probability measure.*

If the entropy $h_\mu(f) = 0$ then conditional measures induced by μ on the leaves of the foliation W are δ-measures, i.e. for almost every leaf the conditional measure is concentrated at a single point.

If $h_\mu(f) > 0$ then conditional measures on almost all leaves are nonatomic.

1.5. Invariant measures for algebraic Anosov actions. All known ergodic invariant measures for algebraic Anosov actions of higher–rank abelian groups can be classified as follows

(1) Haar measures on homogeneous invariant submanifolds;
(2) Lifts of invariant measures from rank one factors on invariant homogeneous submanifolds.

The first case includes the measures concentrated on compact (periodic) orbits; these measures obviously have zero entropy for all elements of the action. Note that the union of such orbits is dense.

Appearance of the second case indicates certain reducibility. While it is reasonable to conjecture that no other ergodic measures exist ([**KS3**], Introduction) current methods allow to deal only with measures which somehow reflect the hyperbolic behavior of the system. The standard assumption would be that a certain element of the action has positive entropy with respect to the measure. This is crucial since our methods are based on considerations of *conditional* measures induced by an invariant measure on stable foliations of various elements of an action as well as on various invariant subfoliations of such foliations. As 1.1 shows such a measure must induce nontrivial (in fact, nonatomic) conditional measures on the leaves of the stable and unstable foliations for any element of the action.

A hyperbolic map has zero entropy with respect to an invariant measure if and only if the conditional measures on its contracting foliation are atomic which happens to be equivalent in this case that these are δ–measures. Under various assumptions we are able to show that an alternative to being atomic for the conditional measure is to be Haar on a certain homogeneous submanifold. This in turn in certain situations leads to the conclusion that the measure itself is of type (1) above.

This approach has been developed in [**KS3**]. The case of actions by automorphisms of the torus (as well as the more general non–invertible situation) is discussed in [**KS3**] in greater detail; the symmetric space actions (Weyl chamber flows and related actions) as well as twisted Weyl chamber flows are also considered. There are gaps in the proofs of the main results both for the toral case (Theorem 5.1) and for the symmetric space case (Theorem 7.1), which were partially corrected in [**KS4**]. In Section 3 we give a complete argument for the proof of the main theorem in the toral case with a properly modified formulation. This theorem yields the rigidity of measure–preserving conjugacies, centralizers and factors. These results are presented in Section 4; [**KKS**] contains applications of these results which use some subtle number–theoretic information. Apart from presenting complete proofs of the above theorem our arguments have an extra value because they elaborate an important method whose uses extend beyond this particular proof. For example, essentially the same method is used in the proof of rigidity of joinings in [**KaK**].

Before doing that in the next section we will explain some of the main features of the method by considering in a less formal way a special case of a $\mathbb{Z}^2$ action on the three–dimensional torus by hyperbolic automorphisms. In this example the underlying geometric structure is quite transparent and a number of complications which appear in more general situations do not show up. In more general cases in order to conclude the desired dichotomy (Haar on a rational subtorus or a homogeneous invariant submanifold, or zero entropy) one needs extra assumptions on the action (e.g. irreducibility in the toral case) or on the measure (such as K-property, or, in the symmetric space case, weak mixing). The conclusion in general is also somewhat weaker.

2. The simplest higher rank models

2.1. Cartan action on the three–dimensional torus. The following special case provides an excellent insight in the core features of the method used to study rigidity of invariant measures.

Let $A \in SL(3,\mathbb{Z})$ be any hyperbolic matrix with distinct real eigenvalues. By passing to a power, if necessary, we may assume that its eigenvalues λ_1, λ_2, λ_3 are all positive. By the Dirichlet unit theorem the centralizer of A in $SL(3,\mathbb{Z})$ is a finite extension of $\mathbb{Z}^2$. See [**KKS**] for a more detailed discussion and [**BS**] for the number–theoretic background. In particular, this centralizer contains a subgroup isomorphic to $\mathbb{Z}^2$ which consist of hyperbolic matrices. Such a subgroup σ determines an action of $\mathbb{Z}^2$ on the three–dimensional torus $\mathbb{T}^3$ by hyperbolic automorphisms which we will denote by ρ. (See Section 1.2) All elements of σ are simultaneously diagonalizable. The eigenvectors define three one–dimensional ρ–invariant linear foliations on the torus, W_1, W_2 and W_3. Each of these foliations possesses the natural affine parameter which is preserved by ρ. This is just another way of saying that the Euclidean length along each foliation is multiplied by each element of the action ρ by a constant which is equal to the absolute value of the corresponding eigenvalue. While no element of the action has an eigenvalue one or minus one there are elements for which the eigenvalue is arbitrary close to one. A useful technical devise is the *suspension* construction which by passing from the torus to a certain five–dimensional solvmanifold (an extension of $\mathbb{T}^2$ by $\mathbb{T}^3$) produces an $\mathbb{R}^2$ action α quite closely reflecting the features of the action ρ. (See Section 1.2.2) In particular, the "fiber" $\mathbb{T}^3$ direction splits into three invariant directions generating invariant foliations which we will still denote by W_1, W_2 and W_3. For each of these foliations there is an *irrational* direction in $\mathbb{R}^2$ such that the action along this direction preserves the length along the foliation. This direction is that of the kernel of the corresponding Lyapunov characteristic exponent, i.e. the Lyapunov hyperplanes defined in 1.2.3 which are of course just lines in our case. Call this direction the *critical* direction for the corresponding foliation. See Section 5.2 below for pertinent definitions in much greater generality.

2.2. Rigidity of invariant measures for Cartan actions. Now consider an ergodic invariant measure for the action ρ. We will show that

($\mathcal{D}$) *Either every element of the action has zero entropy with respect to this measure or the measure is Lebesgue measure on* $\mathbb{T}^3$.

First, extend the measure in a canonical way to an (also ergodic) invariant measure μ of the suspension action α. Let W be one of the three invariant foliations described above. The action preserves the system of conditional measures on the leaves on W. (See Section 1.4). Our argument works if the action in the critical direction is ergodic with respect to μ, or, more generally,

($\mathcal{E}$) *The ergodic components of the action in the critical direction consist of the whole leaves of* W.

This assumption will be later justified for any measure ergodic with respect to ρ. Under assumption ($\mathcal{E}$) we will show that

($\mathcal{T}$) *For almost every leaf* L *of the foliation* W *the conditional measure* μ_L *is invariant under the set of translations of full* μ_L *measure.*

Naturally, translations are defined with respect to the Euclidean length parameter. Here we mean invariance in an exact sense, not up to a scalar multiple. In other words, the measure μ_L is defined up to a scalar multiple but for any choice of normalization the measure is invariant with respect to almost every translation.

2.2.1. Thus L can be identified with the real line. Let us show that property $(\mathcal{T})$ implies that

The measure μ_L is either concentrated in a single point, or is a counting measure concentrated on a certain arithmetic progression, or is invariant under all translations, i.e. is simply Lebesgue measure on the line.

For, the measure is either atomic or continuous (translations cannot mix two parts). In the former case the atoms all have the same measure and hence must be isolated since otherwise the measure would not be locally finite. If there is more than one atom then the support is invariant by *every* translation taking one atom into another, which hence have to form a lattice.

If the measure is nonatomic than the set of translations under which it is invariant contains two rationally independent translations. Any locally finite measure on the line invariant under two rationally independent translations generates a finite measure on the circle (the factor by the first translation) invariant under an irrational translation and is hence Lebesgue.

Notice that ergodicity of the action implies that only one of the possibilities is realized for almost every leaf L.

The lattice case is, in fact, impossible due to the presence of the elements in the actions which expand the length parameter along W. For, any such element would map the set of leaves with a given value of the progression into the set with a strictly bigger value contradicting the Poincaré recurrence theorem.

Now assume that for almost every leaf L the measure μ_L is concentrated in a single point. The foliation W is the strong stable foliation of a certain elements of our action. But this implies that the entropy of those elements is equal to zero. These elements form a convex open set in $\mathbb{R}^2$ (a *Weyl chamber*, see Section 5.2. The inverses of these elements also have zero entropy. Since entropy is a sub–additive function on $\mathbb{R}^2$ ([**H**], Theorem B), it is equal to zero identically on $\mathbb{R}^2$.

If on the other hand, the measures μ_L are Lebesgue then the same is true for the conditional measures on the linear foliation on the torus. A fixed translation along the leaves of such a foliation is the linear flow on the torus which is in our case uniquely ergodic due to the fact that our action is irreducible over rationals (no invariant rational subtori). Since the conditional measures are invariant under translations so is the global measure which produces those conditionals. Hence by unique ergodicity the measure is itself Lebesgue.

An alternative argument does not use unique ergodicity of irrational translations. Instead one notice that any of the three invariant foliations W_1, W_2 and W_3 can play the role of W in the above argument. Since for any of these there is an element of the action for which it is the whole stable foliation, if at least one of the three systems of conditional measures are atomic the entropy of this element and hence of any element of the action vanishes. On the other hand, if all three systems of conditional measures are Lebesgue, then by Fubini Theorem the global measure is also Lebesgue.

2.2.2. Now we will explain why property $(\mathcal{T})$ holds. This is done in two steps.

Step 1. The measures are invariant in the affine sense: the translations are proportional up to a scalar multiple. To that end, let us fix the normalization at μ–almost every point x in such a way that the conditional measure of the interval $I(x)$ of length one on the leaf $L(x)$ centered at x is equal to one. Identifying $I(x)$ with the standard interval $[0,1]$ via the length parameter we obtain a Borel map from our space to the set of probability measures on the unit interval provided with the weak* topology. On a compact subset C of measure arbitrary close to one this map is continuous (Luzin Theorem). A typical compact piece of a leaf intersects such a subset by a set of almost full conditional measure. Now use our ergodicity assumption $(\mathcal{E})$. Starting from a typical point x on a typical leaf L and moving in the critical direction the interval $I(x)$ comes arbitrary close to μ_L almost every point $y \in L$. In particular, if one assumes that both x and y are typical points of one of the continuity sets C described above, one may assert that the returns also appear on the set C. But this implies that in the limit the images of the conditional measures on $I(x)$ weakly converge to the conditional measure on $I(y)$. On the other hand, since we move along the critical direction the interval $I(x)$ is simply translated. Hence, the normalized conditional measure on $I(y)$ coincides with a translation of the normalized conditional measure on $I(x)$.

Step 2. The normalization constant which we will denote $c(x,y)$, is in fact equal to one. This again follows from the Poincaré recurrence. For, obviously, this constant is equivariant with respect to the action: for $a \in \mathbb{R}^2$, $c(\alpha(a)x, \alpha(a)y) = c(x,y)$. Secondly this is cocycle: $c(x,y)c(y,z) = c(x,z)$. The latter condition implies that along a typical leaf the density is exponential with respect to the natural length parameter. Taking an element a for which $\alpha(a)$ contracts the foliation W we see that the exponent must grow contradicting the Poincaré recurrence Theorem again, unless it is zero, *i.e.* the conditional measure is indeed Lebesgue.

2.2.3. Finally, we need to check assumption $(\mathcal{E})$ for any α–invariant ergodic measure, namely to show that the ergodic components for the action in the critical direction consist of the whole leaves of W. To that end we will use the structure of stable and unstable foliations for different elements or the action. In fact, for any of the three foliations there exists an element $a \in \mathbb{R}^2$ such that this foliation is the stable foliation of $\alpha(a)$ and the sum of the other two is the unstable one. On the other hand, the classical Hopf argument shows that ergodic decomposition for any element consists of complete leaves of its stable and unstable foliations: the positive (negative) time average of any continuous function is constant along the stable (unstable) leaves. The remaining ingredient is an important observation that for a normally hyperbolic (generic) element of the action the measurable hulls of partitions into leaves of the stable and unstable foliation coincide, because each of them generates the Pinsker σ–algebra (maximal σ–algebra with zero entropy).

Let us denote by ξ_a the partition into ergodic component of the element $\alpha(a)$ and by $\xi(W)$ the measurable hull of the partition into the leaves of the foliation W. Now let a be a non–zero element in the critical direction, W' be the one–dimensional stable foliation of a, W'' be the remaining foliation and $b \in \mathbb{R}^2$ be a regular (non–critical) element such that W' is the stable foliation of the element $\alpha(b)$. Thus we have the following inequalities:

$$\xi_a \leq \xi(W') = \pi(\alpha(b)) = \xi(W \oplus W'') \leq \xi(W). \tag{2.2.1}$$

2.3. $\times 2$, $\times 3$ and automorphisms of a solenoid. A very similar situation appears for commuting expanding maps of the circle. The basic example is the action of $\mathbb{Z}_+^2$ generated by multiplications by 2 and by 3 (mod 1). The question about invariant measures of this actions was posed by Furstenberg in 1967 [**F**]; it was solved for measures with positive entropy by D. Rudolph, [**R**]. It was an attempt to understand Rudolph's result in a geometric fashion that led the second author to the consideration of the model on the three–torus described above. Now in order to prove that the only ergodic positive entropy invariant measure for the multiplications by 2 and by 3 is Lebesgue we pass to the natural extension for this action. The phase space for this natural extension is a solenoid, the dual group to the discrete group $\mathbb{Z}(1/2, 1/3)$. It is locally modeled on the product of $\mathbb{R}$ with the groups of 2–adic and 3–adic numbers. Thus while topologically the solenoid is one–dimensional, there are three Lyapunov exponents, one for the real direction, and two for the non–Archimedean ones. Since the multiplication by 2 is an isometry in the 3–adic norm and vice versa the critical lines in this case are the two axis and the line $x \log 2 + y \log 3 = 0$ which does not intersect the first quadrant. All the above arguments work in this case verbatim with the real foliation playing the role of W. The unique ergodicity of the flow of translations along the real direction follows form the construction of the solenoid.

This argument of course extends to multiplications by p and q unless for some natural numbers k and l, $p^k = q^l$.

2.4. Other types of rigidity. The models discussed in this section are also very convenient for demonstrating a other types of rigidity phenomena which appear in actions of higher rank abelian groups.

The first type is rigidity of vector valued Hoelder and differentiable cocycles. Rigidity in these cases means that every cocycle from a given class is cohomologous to a constant coefficient cocycle, i.e. a homomorphism from the acting group to the vector space with the cohomology given by a transfer function from the same class or with moderate loss of regularity. See [**KS1**] for 1-cocycles over differentiable actions, [**KSch**] for 1-cocycles over actions by automorphisms of compact groups and [**KK**] for higher-order cocycles. A nice survey which also discusses other types of cocycles, such as those with values with compact abelian groups as well as nonabelian groups is [**NT**].

Another type of rigidity is local differentiable rigidity: any smooth action close to a given action in C^1 topology is differentiably conjugate (also maybe with a small loss of regularity) to the original action; in the continuous case up to an automorphism of $\mathbb{R}^k$ close to identity (see [**KS2**]). It is interesting to point that the proof of local differentiable rigidity involves a construction of a certain family of invariant geometric structures on certain invariant foliations of the perturbed action. In the simplest cases this structure is an affine connection. Then the structural stability implies existence of continuous conjugacy between the original and perturbed action (up to a time change in the continuous case). The principal idea in the proof that the conjugacy must be smooth involves showing that it intertwines a standard structure on the leaves of an invariant foliation for the algebraic action with the corresponding structure of the perturbed one. These latter structure plays a role very similar to that of conditional measures in the setup of the present paper.

There are also global rigidity results which assume that an action has certain topological (e.g. homotopy type) and dynamical (i.e. Anosov) properties and assert conjugacy with a standard algebraic action [**KL, MQ**].

The forthcoming notes [**K2**] discuss all these types of rigidity primarily in the simplest setting of actions of $\mathbb{Z}^2$ by automorphisms of $\mathbb{T}^3$.

3. Rigidity of invariant measures for actions by toral automorphisms

3.1. The Main Theorem and its Corollaries. In this section we discuss the main results on the rigidity of invariant measures for higher rank abelian actions by toral automorphisms. Theorem 3.1 is a modified version of the main result of A. Katok and R. Spatzier for the case of $\mathbb{Z}^k$ action by toral automorphisms (Theorem 5.1' of [**KS4**], which is in turn a modification of Theorem 5.1 of [**KS3**]). We give a complete self-contained proof of this theorem which is based on the original proof in [**KS3**] and [**KS4**].

The modification reflects the new version of Lemma 5.8' from [**KS4**] (see Lemma 3.11). The proof of Lemma 5.8' in [**KS4**] contains a gap since the ergodicity of the action may not imply that subspaces S_x are parallel. This in particular forces a slight modification in the formulation of the result, namely splitting of the measure into finitely many components. We give a more detailed proof of Lemma 3.2 (Lemma 5.4 from [**KS3**]) and a more elementary proof of Lemma 3.4 (Lemma 5.6 from [**KS3**]). Lemma 3.6 and Lemma 3.8 are similar to Lemma 5.9 and Lemma 5.10 in [**KS3**]. and the proofs employ the idea of the proofs in [**KS3**]. While the latter lemmas are correct their proofs in [**KS3**] lack some essential details. We give new arguments to complete the proofs.

Let α and α' be actions of $\mathbb{Z}^k$ by toral automorphisms and let α' be an algebraic factor of α. Then α' is called a *rank–one factor* of α if $\alpha'(\mathbb{Z}^k)$ has a subgroup of finite index which consists of powers of a single map.

For an action α of $\mathbb{Z}^k$ by automorphisms a torus the following two conditions are equivalent [**S**].

$(\mathcal{R})$: The action α contains a subgroup ρ, isomorphic to $\mathbb{Z}^2$, which consists of ergodic automorphisms.

$(\mathcal{R}')$: The action α does not possess non–trivial rank one algebraic factors.

THEOREM 3.1. *Let α be an $\mathbb{R}^k$-action with $k \geq 2$ induced from $\mathbb{Z}^k$ action by automorphisms of $\mathbb{T}^m$ satisfying condition $(\mathcal{R})$. Assume that μ is an ergodic invariant measure for α such that there are generic singular elements $a_1, \ldots, a_k$ and a regular element $b \in \mathbb{R}^k$ such that*

C1: *: $E_b^- = \sum_i (E_{a_i}^0 \cap E_b^-)$ (where the sum need not be direct) and*
C2: *: $\xi_{a_i} \leq \xi(E_{a_i}^0 \cap E_b^-)$.*

Then either the invariant measure $\mu_{\mathbb{T}^m}$ for the $\mathbb{Z}^k$ action by automorphisms of $\mathbb{T}^m$ has zero entropy for all elements of the action or it decomposes as $\mu_{\mathbb{T}^m} = \frac{1}{N}(\mu_1 + \ldots + \mu_N)$, where measures μ_i, $i = 1, \ldots, N$, are invariant under $\alpha(\Gamma)$ for some finite index subgroup $\Gamma \subset \mathbb{Z}^k$. The actions $(\alpha(\Gamma), \mu_i)$ are algebraically isomorphic by the toral automorphisms from $\alpha(\mathbb{Z}^k)$. Each μ_i, $i = 1, \ldots, N$, is an extension of a zero entropy measure in an algebraic factor for $\alpha(\Gamma)$ of smaller dimension with Haar conditional measures in the fibers.

Condition **C2** (which of course depends on **C1** while the latter does not mean much by itself) is a generalization of the condition $(\mathcal{E})$ form Section 2.1. It will play a similar role in the proof. There are various situations where these conditions can be established. It is trivially satisfied if every one-parameter subgroup of the action α is ergodic, or, equivalently if the original $\mathbb{Z}^k$ action by toral automorphisms is weakly mixing.

COROLLARY 3.1. *Let α be a $\mathbb{Z}^k$-action with $k \geq 2$ by automorphisms of a torus satisfying condition $(\mathcal{R})$. Then any weakly mixing invariant measure for α is an extension of a zero entropy measure in an algebraic factor with Haar conditional measures in the fibers.*

The following corollary of Theorem 3.1 will be used in Section 4.

THEOREM 3.2. *Let α be an action of $\mathbb{Z}^2$ by ergodic toral automorphisms and let μ be an α-invariant weakly mixing measure such that for some $\mathbf{m} \in \mathbb{Z}^2$, $\alpha(\mathbf{m})$ is a K-automorphism. Then μ is a translate of Haar measure on an α-invariant rational subtorus.*

PROOF. By Theorem 3.1 the measure μ is an extension of a zero entropy measure for an algebraic factor of smaller dimension with Haar conditional measures in the fiber. Since α contains a K-automorphism it does not have nontrivial zero entropy factors. Hence the factor in question is the action on a single point and μ itself is a Haar measure on a rational subtorus. □

In the special case considered in Section 2.2 the ergodicity assumption $(\mathcal{E})$ for the critical direction was deduced from ergodicity. The deduction was based on the interlacing of stable and unstable foliations which allowed to produce (2.2.1). Now we will show how a similar argument allows to verify conditions of Theorem 3.1 in much greater generality.

THEOREM 3.3. *Any ergodic invariant measure μ for a totally nonsymplectic Anosov action of $\mathbb{Z}^k$ by automorphisms of a torus either has zero entropy for all elements of the action or decomposes as $\mu = \frac{1}{N}(\mu_1 + ... + \mu_N)$, where measures μ_i, $i = 1, ..., N$, are invariant under $\alpha(\Gamma)$ for some finite index subgroup $\Gamma \subset \mathbb{Z}^k$. The actions $(\alpha(\Gamma), \mu_i)$ are algebraically isomorphic by the toral automorphisms from $\alpha(\mathbb{Z}^k)$. Each μ_i, $i = 1, ..., N$, is an extension of a zero entropy measure in an algebraic factor for $\alpha(\Gamma)$ of smaller dimension with Haar conditional measures in the fibers.*

PROOF. First for any Anosov TNS action of $\mathbb{Z}^k$, $k \geq 2$, since for $k = 1$ any positive exponent is negatively proportional to any negative exponent. Thus we need to check conditions **C1, C2**. For that it is obviously sufficient to show that for any generic singular element $a \in \mathbb{R}^k$, $\xi_a \leq \xi(E^0_a)$.

For such an element a a single Lyapunov exponent vanishes and the corresponding Lyapunov distribution coincides with E^0_a. Take a regular element b nearby for which the corresponding Lyapunov exponent is positive and all other exponents have the same signs as for a. Thus $E^+_b = E^+_a \oplus E^0_a$ and $E^-_b = E^-_a$. Birkhoff averages with respect to $\alpha(a)$ of any continuous function are constant on the leaves of ξ_a. Since such averages generate the algebra of $\alpha(a)$ invariant functions we conclude that $\xi_a \leq \xi(E^-_a)$. On the other hand both and both $\xi(E^-_b)$ and $\xi(E^+_b)$ coincide with the Pinsker algebra $\pi(\alpha(b))$. Thus we conclude

$$\xi_a \le \xi(E_a^-) = \xi(E_b^-) = \pi(\alpha(b)) = \xi(E_b^+) = \xi(E_a^+ \oplus E_a^0) \le \xi(E_a^0).$$

□

COROLLARY 3.2. *Any ergodic invariant measure for an irreducible totally non-symplectic Anosov action of $\mathbb{Z}^k$ by automorphisms of a torus either has zero entropy for all elements of the action or is Lebesgue measure on the torus.*

3.2. Scheme of the Proof of Theorem 3.1. *Step 1.* Let us consider one of the generic singular elements a_i from the statement of Theorem 3.1 and a Lyapunov exponent λ such that $\lambda(a_i) = 0$. We denote by F the invariant foliation $W^I_{a_i} \cap W^-_b \cap W^\lambda$ where W^λ is the Lyapunov foliation corresponding to λ.

In Section 3.3 we use Lemmas 3.2, 3.3, 3.4 to establish the following dichotomy.

LEMMA 3.1. *For any foliation F described above either*

(1) *The conditional measure μ_x^F is atomic for μ-a.e. x, or*
(2) *The conditional measure μ_x^F is a Haar measure on an affine subspace S_x of positive dimension for μ-a.e. x.*

Step 2. If for all foliations F described in *Step 1* the first alternative of Lemma 3.1 takes place then we prove in Section 3.4 that the conditional measures on the foliation W^-_b are atomic. By Proposition 1.1 this implies that the entropy of $\alpha(b)$ is equal to 0. Since the stable foliation is the same for all elements in the same Weyl chamber we see that the entropy is 0 for all elements in the same Weyl chamber as b. Since the entropies of $\alpha(b)$ and $\alpha(-b)$ are the same we conclude the entropy is 0 for all elements in the Weyl chamber of b. Then it follows from sublinearity of entropy [**H**] that all elements of the action have 0 entropy.

This completes the proof of Theorem 3.1 in the case when the first alternative of Lemma 3.1 takes place for all foliations F described in Step 1.

Step 3. Suppose that for some foliation F described in *Step 1* the second alternative of Lemma 3.1 takes place, i.e. the conditional measure μ_x^F is a Haar measure on an affine subspace S_x for μ-a.e. x. We note that by ergodicity of the action the subspaces S_x have the same dimension for μ-a.e. x. We may assume that this dimension is positive. Then Lemma 3.11 shows that the measure on the torus decomposes as $\mu_{\mathbb{T}^m} = \frac{1}{N}(\mu_1 + ... + \mu_N)$, where measures μ_i, $i = 1, ..., N$, are invariant under $\alpha(\Gamma)$ for some finite index subgroup $\Gamma \subset \mathbb{Z}^k$. The actions $(\alpha(\Gamma), \mu_i)$ are algebraically isomorphic and each μ_i, $i = 1, ..., N$, is an extension of a zero entropy measure in an algebraic factor of smaller dimension for $\alpha(\Gamma)$ with Haar conditional measures in the fibers. We restrict the action α to the finite index subgroup Γ and consider invariant measure μ_1. If the factor–measure has zero entropy for all elements of this action we are done. Otherwise we may assume that in the factor some element still has positive entropy and repeat the argument. Note that condition $(\mathcal{R})$ is inherited by any algebraic factor as well as conditions **C1, C2**. We will arrive at a factor of the factor and so on. Since at every step the dimension of the factor drops this process has to stop, thus producing a factor with zero entropy. It is also clear by induction that the conditionals are in fact Haar measures on the fibers.

Notice that in the case of $\mathbb{Z}^2$ action on $\mathbb{T}^3$ considered in Section 2.2 Steps 2 and 3 are not needed.

3.3. Proof of the Lemma 3.1. In this section we prove the following three lemmas from [**KS3**] which establish the dichotomy of Lemma 3.1 from Section 3.2. Lemma 3.2 generalizes the argument from the Step 1 in Section 2.2.2, and Lemma 3.4 the argument from Step 2 in the same section.

The general setup for these lemmas is as follows. Suppose that $a \in \mathbb{R}^k$ is a generic singular element and $F \subset W_a^I$ is some $\alpha(a)$-invariant subfoliation of W_a^I with leaves of dimension d. Denote by $B_1^F(x)$ the closed unit ball in $F(x)$ about x with respect to the flat metric. Denote by μ_x^F the system of conditional measures on F normalized by the requirement $\mu_x(B_1^F(x)) = 1$ for all x in the support of μ and by G_x the subgroup of isometries of $F(x)$ which preserve μ_x^F up to a scalar multiple.

To simplify the notation from now on we will write a instead of $\alpha(a)$ when this does not cause any confusion.

Recall that we denote by ξ_a the partition into ergodic components of element a and by $\xi(F)$ the measurable hull of F.

LEMMA 3.2. *([**KS3**], Lemma 5.4) Suppose that $\xi_a \leq \xi(F)$. Then for μ-a.e. x, G_x is closed and the support of μ_x^F is the orbit of x under the group G_x . Furthermore, $\phi_*\mu_x^F = \mu_{\phi x}^F$ for any $\phi \in G_x$.*

PROOF. To prove the last statement of the lemma we note that the normalizations of the conditional measures $\phi_*\mu_x^F$ and μ_y^F coincide since $\phi_*\mu_x^F(B_1^F(y) = \mu_x^F(B_1^F(x)) = 1$ due to the fact that ϕ is an isometry.

Since G_x maps the support of μ_x^F to itself we only need to show that G_x is closed and acts transitively on the support of μ_x^F.

We first show that G_x is closed. Let $\{\phi_n\} \subset G_x$ be a sequence of isometries converging to an isometry ϕ. We need to show that $\phi \in G_x$. Let $y_n = \phi_n(x)$ and $y = \phi(x)$. We can choose a radius r such that the boundaries of balls $B_r^F(x)$ and $B_r^F(y)$ carry no conditional measure. Let us denote by $\mu_z^{F,r}$ the conditional measure normalized by $\mu_z^{F,r}(B_r^F(z)) = 1$. Since the balls $B_r^F(y_n)$ converge to $B_r^F(y)$ we conclude that $\mu_{y_n}^{F,r}$ converge to $\mu_y^{F,r}$ in weak* topology. Since $\phi_n \in G_x$ we see that $(\phi_n)_*\mu_x^{F,r} = \mu_{y_n}^{F,r}$ since their normalizations coincide. Since $(\phi)_*\mu_x^{F,r}$ is the weak* limit of $(\phi_n)_*\mu_x^{F,r}$ it follows that $(\phi)_*\mu_x^{F,r} = \mu_y^{F,r}$. Since $\mu_y^{F,r}$ is a scalar multiple of $\mu_x^{F,r}$, we conclude that ϕ belongs to G_x.

It remains to show that for μ-a.e. x and μ_x^F-a.e. $y \in F(x)$ there exists an isometry ϕ of $F(x)$ such that $y = \phi(x)$ and $\mu_y^F = \phi_*\mu_x^F$. Since G_x is closed and any set of full measure is dense in the support of μ_x^F, it would follow that G_x is, in fact, transitive on the support of μ_x^F.

Let X_1 be the set of all x such that the ergodic component E_x of a passing through x is well-defined and contains $F(x)$ (up to a set of μ_x^F-measure 0). By the assumption on a, the set X_1 has full μ-measure. Let μ_x be the induced measure on E_x. Recall that a acts isometrically on $F(x)$.

Let us fix $R > 0$ and introduce the following notation

$$\mu_{x,R}^F = \mu_x^F|_{B_R^F(x)}.$$

We can canonically identify each ball $B_R^F(x)$ with the standard ball B_R in $\mathbb{R}^{\dim F}$ of radius R. Thus we can consider the system of measures $\mu_{x,R}^F$ as a measurable function from M to the weak* compact set of Borel probability measures on B_R (compare with the proof of $(\mathcal{T})$, step 1 in Section 2.2). By Luzin's theorem, we can

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take an increasing sequence of closed sets K_i contained in the support of μ such that

(1) $\mu(K) = 1$, where $K = \bigcup_{i=1}^{\infty} K_i$

(2) $\mu^F_{x,R}$ depends continuously on $x \in K_i$ with respect to the weak* topology.

Set $X_2 = X_1 \cap K$. Since the transformation induced by a on $X_2 \cap E_x$ is ergodic, the transformation induced by a on $X_1 \cap K_i \cap E_x$ is also ergodic for any i. Hence the set X_3, which consists of points $x \in X_2$ whose orbit $\{a^n x\}_{n\in\mathbb{Z}}$ is dense in $X_1 \cap K_i \cap E_x$ for all i, has full μ-measure.

Let $x \in X_3$ and $y \in X_3 \cap F(x)$. Then $x, y \in X_1 \cap K_i \cap E_x$ for some i. Hence there exists a sequence $n_k \to \infty$ such that the points $y_k = a^{n_k} x \in X_1 \cap K_i \cap E_x$ converge to y. Since $\mu^F_{x,R}$ depends continuously on $x \in K_i$ with respect to the weak* topology the measures $\mu^F_{y_k,R}$ weak* converge to the measure $\mu^F_{y,R}$. Let us denote the isometry $a^{n_k}|_{F(x)}$ by ϕ_k. We note that $(\phi_k)_*\mu^F_{x,R} = \mu^F_{y_k,R}$ since they are conditional measures on the same leaf $F(y_k)$ and their normalizations coincide. Taking a subsequence if necessary, we may assume that ϕ_k converge uniformly on compact sets to some isometry ϕ of $F(x)$ with $\phi(x) = y$. Since $(\phi)_*\mu^F_{x,R}$ is the weak* limit of $(\phi_k)_*\mu^F_{x,R}$ it follows that $(\phi)_*\mu^F_{x,R} = \mu_{y,R}$.

We conclude that for any $R > 0$ there exists a set X_3 of full μ measure such that for any $x \in X_3$ and $y \in X_3 \cap F(x)$ there exists an isometry ϕ of $F(x)$ with the properties $\phi(x) = y$ and $\mu^F_{y,R} = \phi_* \mu^F_{x,R}$.

We can now take a sequence $R_i \to \infty$ and choose a set X of full μ-measure such that for any $x \in X$, $y \in X \cap F(x)$ and any i there exists an isometry ϕ_i of $F(x)$ with the property $\mu^F_{y,R_i} = (\phi_i)_* \mu^F_{x,R_i}$. Taking a converging subsequence we obtain that for any $x \in X$ and $y \in X \cap F(x)$ there exists an isometry ϕ of $F(x)$ with the properties $\phi(x) = y$ and $\mu^F_y = \phi_* \mu^F_x$. This completes the proof of the lemma since we may assume that the set X is chosen so that for $x \in X$ the set $X \cap F(x)$ has full μ^F_x-measure. □

LEMMA 3.3. *([**KS3**], Lemma 5.5) In addition to the assumptions in Lemma 3.2, let F be contained in the intersection of W^I_a with a Lyapunov subspace for a non-zero Lyapunov exponent λ. Then for μ-a.e. x, the support S_x of μ^F_x is an affine subspace of $F(x)$ whose dimension is constant for almost every x..*

PROOF. By Lemma 3.2, S_x is the orbit of a closed group of isometries. Therefore S_x is a submanifold, possibly disconnected. Note that the maximal principal curvature of S_x is constant along S_x. Let $\kappa(x)$ denote this constant.

Let b be any element such that $\lambda(b) < 0$. Note that b maps S_x to $S_{b\,x}$. Iterates of b exponentially contract the fibers of F. In particular, since the exponential contractions in all directions inside F are the same, any curve with positive principal curvature will be mapped to curves with exponentially increasing principal curvatures. Hence $\kappa(b^n x)$ goes to infinity for μ-a.e. x unless $\kappa(x) = 0$. This is impossible by Poincaré recurrence. Thus $\kappa(x) \equiv 0$, and hence the support of μ^F_x is a union of non-intersecting affine subspaces.

Let us now show that the support is connected. Suppose to the contrary that the support is a union $\cup A_i$ of at least two affine subspaces A_i. Let d_x denote the minimum of the distances from x to any A_i which does not contain x. Since the support is a closed subset, $d_x > 0$ for all x. Note that $d_{b^n\,x} \to 0$ as $n \to \infty$. This

is again a contradiction to Poincaré recurrence. The dimension of S_x is α-invariant and hence is constant due to the ergodicity of the action α. □

LEMMA 3.4. *([**KS3**], Lemma 5.6) Under the assumptions of Lemma 3.3, μ_x^F is Haar measure on S_x.*

PROOF. By Lemma 3.2 the group G_x of isometries of $F(x)$ which map μ_x^F to its scalar multiple acts transitively on the support S_x of the conditional measure of μ_x^F. By Lemma 3.3, S_x is an affine space. For any x and $y \in S_x \subset F(x)$ let us define the scaling coefficient $c_x(y)$ by the equality $\phi\mu^F = c_x(y)\mu^F$, where μ^F is the conditional measure on $F(x) = F(y)$ and $\phi \in G_x$ is an isometry such that $\phi(x) = y$. In other words $c_x(y)$ can be calculated as

$$c_x(y) = \frac{\phi_*\mu^F(A)}{\mu^F(A)} = \frac{\mu^F(\phi^{-1}A)}{\mu^F(A)} = \frac{\mu^F(A)}{\mu^F(\phi A)}$$

for any set A of positive conditional measure. We note that since the conditional measures are defined up to a scalar multiple it is clear that the definition does not depend on a particular choice of μ^F. Since the image of the unit ball $\phi(B_1^F(x))$ is the same for all isometries $\phi \in G_x$ such that $\phi(x) = y$ we conclude that the definition does not depend on a particular choice of ϕ either.

Since we can take the test set A such that the conditional measure of the relative (to $F(x)$) boundary of ϕA is equal to zero, we conclude that that for a fixed x the coefficient $c_x(y)$ depends continuously on y.

We see that either for $\mu - a.e.\ x\ \ c_x(y) = 1$ for all $y \in S_x$, hence μ_x^F is Haar on S_x, or there exists a set X of positive measure such that $c_x(y)$ is not identically equal to 1 for $x \in X$. In the latter case for some $\epsilon > 0$ we can define a finite positive measurable function

$$f_\epsilon(x) = \inf\{r : \exists\ y \in S_x\ s.t.\ d(x,y) < r\ and\ |c_x(y) - 1| > \epsilon\}$$

on some subset $Y \subset X$ of positive μ-measure. By measurability there exists N and a set Z of positive measure on which f_ϵ takes values in the interval $(1/N, N)$. We will show that

$$f_\epsilon(b^n x) \to 0 \text{ as } n \to \infty \tag{3.3.1}$$

uniformly on Z for an element b which contracts foliation F. This will prove the lemma since it contradicts to the recurrence of Z under b.

We will prove now that

$$c_x(y) = c_{bx}(by) \tag{3.3.2}$$

for μ-a.e. x and $y \in S_x$. Since the iterates of b exponentially contract the leaves of F this invariance property implies that $f_\epsilon(b^n x) \le C\lambda^n f_\epsilon(x)$, for some $C,\lambda > 0$, hence (3.3.2) implies (3.3.1)

To prove (3.3.2) we use Lemma 3.2. For any $y \in S_x$ there exists an isometry ϕ of $F(x)$ with the properties $\phi(x) = y$ and $\mu_y^F = (\phi)_*\,\mu_x^F$. Since such ϕ can be obtained as a limit of a sequence of some powers of a restricted to $F(x)$ we may assume that it commutes with b in the following sense. The map $\psi = b \circ \phi \circ b^{-1}$ is an isometry of $F(bx)$. Thus we have $b \circ \phi = \psi \circ b$. We note that since b preserves the family of conditional measures (up to a scalar multiple) ψ preserves the conditional

measures on $F(bx)$ up to a scalar multiple. Hence for any set $A \subset F(x)$ of positive conditional measure we have

$$c_x(y) = \frac{\mu^F(A)}{\mu^F(\phi A)} = \frac{(b_*\mu^F)(bA)}{(b_*\mu^F)(b \circ \phi A)}$$
$$= \frac{(b_*\mu^F)(bA)}{(b_*\mu^F)(\psi \circ bA)} = \frac{(b_*\mu^F)(bA)}{(b_*\mu^F)(\psi(bA))} = c_{bx}(by).$$

□

3.4. The Case of Atomic Conditional Measures. In this section we consider the case when for all foliations F, described in *Step 1* of the scheme of the proof, the first alternative of Lemma 3.1 takes place, i.e. the conditional measures are atomic. We use Lemmas 3.5, 3.6, and 3.8 to show that in this case the conditional measures on the foliation W_b^- are atomic. This by Proposition 1.1 implies that the entropy of $\alpha(b)$ is equal to 0 and allows us to proceed as in *Step 2* of the scheme of the proof.

REMARK 3.1. *This part of the argument is not needed in the case considered in Section 2.2 since the foliation F in that case coincides with the complete stable foliation of some regular element b and hence zero entropy follows right away.*

As in *Step 1* of the scheme of the proof of Theorem 3.1 we consider one of the generic singular elements a_i from the statement of Theorem 3.1 and a Lyapunov exponent λ such that $\lambda(a_i) = 0$. Let us denote by F the invariant foliation $W_{a_i}^I \cap W_b^- \cap W^\lambda$ where W^λ is the Lyapunov foliation corresponding to λ. We know that the conditional measure μ_x^F is atomic for μ-a.e. x.

First we use Lemma 3.5 to conclude that the conditional measures on the foliation $W_{a_i}^I \cap W_b^-$ are atomic for each a_i. We note that Lemma 3.2 implies that the conditional measures on the whole foliation $W_{a_i}^I \cap W_b^-$ are supported on smooth submanifold of the leaves.

Once we know that the conditional measures on foliations $W_{a_i}^I \cap W_b^-$ for all i are atomic Lemma 3.6 shows that the conditional measures on all foliations $W_{a_i}^0 \bigcap W_b^-$ are also atomic. Then to show that the conditional measures on the whole W_b^- are atomic we use the inductive process as in the end of the outline of the proof of Theorem 5.1 in [**KS3**].

We restrict the action to a 2-plane which contains b and intersects all the Lyapunov hyperplanes in generic lines. Then we can replace each a_i by an element c_i in the intersection of the 2-plane with the unique Lyapunov hyperplane that contains a_i. Since the elements a_i and c_i have the same center foliation for each i we see that

$$E_b^- = \sum_i (E_{a_i}^0 \cap E_b^-) = \sum_i (E_{c_i}^0 \cap E_b^-)$$

and the conditional measures on all $W_{c_i}^0 \cap W_b^-$ are atomic. We can now reorder the elements c_i in such a way that the Lyapunov exponent that correspond to $E_{c_j}^0 \cap E_b^-$ is negative on c_i for each $j < i$. Starting with $W_{c_1}^0 \cap W_b^-$ we apply Lemma 3.8 inductively to prove that the conditional measures are atomic on $(W_{c_1}^0 \oplus W_{c_2}^0) \cap W_b^-$, on $(W_{c_1}^0 \oplus W_{c_2}^0 \oplus W_{c_3}^0) \cap W_b^-$, and so on until we exhaust the whole W_b^-.

To complete the proof that the conditional measures on the foliation W_b^- are atomic we need to prove Lemmas 3.5, 3.6, and 3.8 below.

The proof of Lemma 3.5 follows the proof of Lemma 5.8 in [**KS3**]. Lemma 3.6 is identical to Lemma 5.9 in [**KS3**] and Lemma 3.8 is a slight modification of Lemma 5.10 from [**KS3**].

Notice that any α-invariant subfoliation of W_a^I splits into its intersections with Lyapunov foliations.

LEMMA 3.5. *Let F be an invariant subfoliation of $W_a^I \cap W_b^-$ and let $F = \sum_\lambda (F \cap W^\lambda)$ be the splitting into its intersections with the Lyapunov foliations. Assume that the conditional measures on all foliations $F \cap W^\lambda$ are atomic and the support S_x of measure μ_x^F is a smooth submanifold of $F(x)$. Then the conditional measures on F are also atomic.*

PROOF. The support S_x of measure μ_x^F is a smooth submanifold which intersects every $F \cap W^\lambda$ in at most one point. Let λ be the Lyapunov exponent smallest on b. Let D be the distribution of tangent spaces of S_x. It is measurable, and b-invariant and C^∞ on $F(x)$. Since D cannot intersect the component in the W_λ-direction in a subspace of positive dimension, D must be tangent to the sum $\sum_{\mu \neq \lambda} W_\mu$ by b-invariance. By taking the Lyapunov exponents inductively in increasing order, we see that D is trivial and μ_x^F are atomic. □

LEMMA 3.6. *Let $a \in \mathbb{R}^k$ be a generic singular element and $b \in \mathbb{R}^k$ be a regular element. If the conditional measures on the foliation $W_a^I \cap W_b^-$ are atomic, then the conditional measures on the foliation $W_a^0 \cap W_b^-$ are also atomic.*

PROOF. Let us introduce the following notations for the proof of the lemma:

$$E = E_a^0 \cap E_b^- \quad , \qquad E^I = E_a^I \cap E_b^- .$$

E^I is the subspace of E spanned by the isometric directions of a, i.e. the eigendirections for real eigenvalues and invariant 2-planes for pairs of complex eigenvalues. Note that E and E^I are both a and b invariant. The corresponding foliations will be denoted by

$$F = W_a^0 \cap W_b^- \quad \text{and} \quad F^I = W_a^I \cap W_b^-$$

correspondingly.

We will prove the lemma by showing inductively that the conditional measures are atomic on the foliations that correspond to larger and larger subfoliations of F, until we exhaust the whole F.

The proof consists of two parts. One part is the basic inductive step which shows that if we add a certain one dimensional foliation to the previously constructed foliation than the conditional measures will be again atomic. This part is established in Lemma 3.7. The other part is the inductive process itself which we explain below. In this part we choose the one dimensional directions and the proper ordering of them to ensure our ability to make the inductive step. This choice is based on the relation between the algebraic properties of elements a and b as linear transformations of E.

The a and b invariant subspace E splits as the direct sum of its intersections with the root subspaces (the generalized eigenspaces) of a:

$$E = \bigoplus_{\lambda \in Sp(a)} Ker(a - \lambda)^m .$$

Each term of this direct sum is b invariant since b commutes with a. Hence it can be split into the direct sum of root subspaces for the restriction of b. Thus we

obtain the splitting $E = \bigoplus U_i$ which is invariant under both elements a and b. Let us denote by λ_i (correspondingly ν_i) the eigenvalue of a (correspondingly b) on the subspace U_i. We have that $|\lambda_i| = 1$ and $|\nu_i| < 1$ for all i. Each U_i is filtered as

$$U_i \cap E^I = V_i^0 \subset ... \subset V_i^{d_i} = U_i \text{ , where } \quad V_i^k = Ker(a - \lambda_i)^{k+1} \cap U_i$$

and $(d_i + 1)$ is the maximal dimension of Jordan blocks of a on U_i. Note that since b commutes with a it leaves each subspace V_i^k invariant. So we see that each V_i^k is both a and b invariant. We assign the "weight" $|\nu_i|^{\frac{1}{j}}$ to each subspace V_i^j, we assign 0 weight to subspace V_i^0. Let us enumerate the weights in the increasing order: $0 = w_0 < w_1 < ... < w_{j_0}$. We denote by E_j the direct sum of all V_i^k whose weights equal to w_j. Now E filtrates as

$$E^I = E_0 \subset ... \subset E_{j_0} = E.$$

Even though the subspaces V_i^k that correspond to the complex eigenvalues are defined only over $\mathbb{C}$ all subspaces E_j are defined over $\mathbb{R}$. This follows from the structure of the Jordan normal form for a over reals. We note again that each E_j is a and b invariant.

We will show inductively that the conditional measures on the foliations corresponding to E_j are atomic.

Let us fix j and assume inductively that the conditional measures on the foliation corresponding to E_{j-1} are atomic. We will prove that the conditional measures on the foliation corresponding to E_j are also atomic by showing that the conditional measures are atomic on the foliations corresponding to larger and larger subspaces of E_j. The dimension of the subspace will be increased by one at each step until we exhaust the whole E_j.

When we would like to add one direction in some U_i for which λ_i or ν_i is not real, instead of elements a and b we will use their suitable powers a^{s_i} and b^{t_i} s.t. $\lambda_i^{s_i} = 1$ and $0 < \nu_i^{t_i} < 1$. By taking the square of the element, if necessary, we may assume that there exist real logarithms and hence real powers of a and b. The ergodicity of these elements is not needed since we only will be using the fact that they preserve the measure. Note that the splitting $E = \bigoplus U_i$ and filtrations $U_i \cap E^I = V_i^0 \subset ... \subset V_i^{d_i} = U_i$ are the same for all such powers. Hence the order of weights and filtration $E^I = E_0 \subset ... \subset E_{j_0} = E$ are also the same.

E_j splits into its intersections with U_i as $E_j = \bigoplus V_i^{k_i}$. Then the intersections of E_{j-1} with U_i are either $V_i^{k_i}$ or $V_i^{k_i-1}$. We will consider only the intersections of E_j with U_i of the second type since we do not need to add any directions along the other U_i's. The operator $(b^{t_i} - \nu_i^{t_i})$ is nilpotent on $V_i^{k_i}/V_i^{k_i-1}$ since it is nilpotent on U_i. Let $d_i^1, ..., d_{i,P_i}$ be the dimensions of the cyclic subspaces of $(b^{t_i} - \nu_i^{t_i})$ on $V_i^{k_i}/V_i^{k_i-1}$ and $e_{i,1}, ..., e_{i,P_i} \in V_i^{k_i}$ be some representatives of their generators. Then the vectors

$$e_{i,p}^l := (b^{t_i} - \nu_i^{t_i})^{d_i^p - (l+1)} e_{i,p} \quad for \quad p = 1, ..., P_i \quad and \quad l = 0, ..., d_i^p - 1$$

form a basis of $V_i^{k_i}$ relative to $V_i^{k_i-1}$. Finally we have

$$E_j = E_{j-1} \oplus \bigoplus_i \bigoplus_{p=1}^{P_i} \bigoplus_{l=0}^{d_i^p - 1} < e_{i,p}^l >$$

We will add the directions $< e^l_{i,p} >$ to E_{j-1} one by one in the order of increasing of the value of $\frac{l}{k_i}$. The rest of the proof describes the procedure of adding.

Let $D =< e^{l_0}_{i_0,p_0} >$ be the new direction to be added at this step. We introduce the following notations for this step: $k_0 := k_{i_0}$, $e_l := e^l_{i_0,p_0}$ for $l = 0, ..., l_0$, $A := a^{s_{i_0}}$ and $B := b^{t_{i_0}}$. The eigenvalues of A and B on U_{i_0} are correspondingly 1 and ν, where $0 < \nu < 1$. The vectors $e^k_l := (A-1)^{k_0-k} e_l$ for $l = 0, ..., l_0$ and $k = 0, ..., k_0$ form a Jordan normal basis for A on some subspace $V \subset U_i$ which will be of particular interest for this step. Note that the subspaces U_{i_0} and V are invariant for A and B as subspaces over $\mathbb{R}$.

Let us denote by D_1 the A and B invariant subspace which is the sum of E_{j-1} and previously added directions. We denote the corresponding foliation by F_1. We assume inductively that the conditional measures on the foliation F_1 are atomic. It is easy to see that the subspace $D_2 = D_1 \oplus D$ is also defined over $\mathbb{R}$ and both A and B invariant. Let us denote the corresponding foliation by F_2.

To complete the proof Lemma 3.6 it remains to prove the following lemma which shows that the conditional measures on the foliation F_2 are atomic.

LEMMA 3.7. *If the conditional measures on the foliation F_1 are atomic then the conditional measures on the foliation F_2 are also atomic.*

PROOF. Let us consider a measurable partition, subordinate to F_2, which consists "mainly" of small "rectangles" of the same size with sides parallel to the basis directions and has the following property. The measure of the set Int_γ is at least 0.99 for some $\gamma > 0$, where Int_γ consists of points inside rectangles on the distance at least γ from the relative (to the leaf of F_2) boundary of the rectangle that contains the point.

By induction hypothesis there exists a set of full measure which intersects any fiber of F_1 in at most one point. Approximating this set from inside we can find a compact set K with the same property. We would like to consider only "good" part of the measure μ, so we introduce a new measure μ_X by $\mu_X(.) = \mu(. \cap X)$, where $X = K \cap Int_\gamma$ with $\mu(X) \geq 0.98$. Let us consider the system of the conditional measures of μ_X w.r.t. the measurable partition into the rectangles (the remaining elements have μ_X measure 0). These measures will be referred to as the conditional measures of rectangles. We will regard each rectangle as a direct product of its vertical (F_1) and horizontal (D) directions. We observe the following dichotomy:

(1) Either for every rectangle, in a set of positive measure, at least 1/3 of its conditional measure is concentrated on a single vertical leaf, hence at one point,
(2) Or there exists a lower bound $d > 0$ for the width of a vertical strip of a rectangle that can carry at least 1/3 of its conditional measure, for any rectangle in a set Y consisting of whole rectangles with $\mu_x(Y) > 0.97$

In the first case the existence of atoms for the conditional measures of rectangles implies the existence of atoms for the conditional measures on foliation F_2. Since this foliation is contracted by b the existence of atoms forces the conditional measures on F_2 be atomic. This can be seen as in the proof of Proposition 4.1 in [**KS3**]. If x is an atom of the conditional measure then there exists a small neighborhood U of x in the leaf $F_2(x)$ such that $\mu^{F_2}_x(U - \{x\}) < \varepsilon\, \mu^{F_2}_x(\{x\})$. Pushing $\mu^{F_2}_x$ backward and using Poincaré recurrence, we see that for a typical x, $\mu^{F_2}_x$ is concentrated at x.

In the latter case each rectangle in Y can be split into three vertical strips of width at least d so that both the left and the right ones have the conditional measure at least $1/3$. We again may assume that the conditional measures do not have atoms since otherwise we could argue as in the first case.

The intersection of the compact set K with any rectangle is a graph of a continuous (not necessarily everywhere defined) function from the vertical direction to horizontal. Moreover the family of these functions is equicontinuous. Let us take some ϵ and find δ given by the equicontinuity. Combining contraction provided by B and shear provided by A we will find such an element $A^m B^n$ that the image of any rectangle from Y has the following properties:

(1) The image is δ-narrow in the horizontal direction
(2) The distance along the vertical direction between the images of the right and the left strips is at least ϵ
(3) The image is sufficiently small (its diameter less than γ) so that it can not intersect γ-interiors of two different rectangles simultaneously.

Under these conditions the images of the right and the left strips can not intersect $Z = X \cap Y$ simultaneously. But this implies that $\mu_X(A^m B^n(Z) \cap Z) \leq \frac{2}{3}\mu_X(Z)$ which is impossible since $\mu(Z) = \mu_X(Z) = \mu_X(Y) \geq 0.97$. To prove that the conditional measures on F_2 are atomic it remains to show that such an element $A^m B^n$ exists.

To satisfy the second condition we obtain the required shear along the direction of the eigenvector e_0^0. On subspace V element B provides uniform contraction by factor ν and possibly some shear. Since no shear accumulates in V along $D =< e_{l_0} >$ direction the size along D direction of the image under B^n of any rectangle is $\nu^n s$, where s is the size of the original rectangle. Since with respect to B vector e_{l_0} has height l_0 over e_0 the distance between the images under B^n of the right and left strips along e_0 direction is at least $c_0 n^{l_0} \nu^n d$ for some small $c_0 > 0$ and sufficiently large n. After this we apply A^m, where m will be exponential in n. We see that the size along D direction is still $\nu^n s$. Since vector e_0 has height k_0 (with respect to A) the leading term of the distance between the images of the right and left strips under $A^m B^n$ along the direction of the eigenvector e_0^0 is $c_1 m^{k_0} n^{l_0} \nu^n d$. Indeed, the other terms with the same power of m have smaller power of n while terms with smaller power of m are negligible since m will be exponential in n. We see that the other terms are bounded by $C_2 m^{k_0} n^{l_0 - 1} \nu^n s$, where C_2 is large (depending only on the structure of the subspace V for A and B), so the distance between the images of the right and left strips along e_0^0 direction is at least $c_3 m^{k_0} n^{l_0} \nu^n d$ for sufficiently large m and n and some small c_3 (depending only on k_0 and l_0). Similarly, the size of the image along all V directions can be estimated from above by $C_4\, m^{k_0} n^{l_0} \nu^n s$.

We would like to show that we can control the size of the image along other directions in E_1 as effectively as along the V directions. For each i the intersection $D_1 \cap U_i$ splits further into A and B invariant real subspaces that can be constructed similarly to V starting from some $e_{i,p}^l \in D_1$ such that $e_{i,p}^{l+1} \notin D_1$.

Let us consider any one of these A and B invariant real subspaces and denote it by $V' \subset D_1 \cap U_i$. As above, we can estimate the size of the image along V' by $C_5\, m^{k_i} n_*^l \nu_*^n s$, where ν_* is the corresponding eigenvalue of b.

We will now specify the choice of the element $A^m B^n$. First we note that the size of the image of any rectangle along D direction is $\nu^n s$ independently on m.

Hence the condition (1) will be automatically satisfied once n is chosen sufficiently large.

We observe that the ratio of the desired shear and the size along V directions is bounded. To make the shear and the size along V directions bounded for large m and n we take m and n so that $m \sim (n^{l_0}\nu^n)^{-\frac{1}{k_0}}$. In this case we will obtain the following estimate of the size of image along the subspace V':

$$C_6(\nu_*^{\frac{1}{k_i}}/\nu^{\frac{1}{k_0}})^{k_i n} n^{k_0(\frac{l}{k_i}-\frac{l_0}{k_0})s}.$$

We observe that this estimate is also bounded since $\nu_*^{\frac{1}{k_i}} = \nu^{\frac{1}{k_0}}$ and $\frac{l}{k_i} < \frac{l_0}{k_0}$ due to the the ordering of weights and the order of adding directions inside E_j. Note that if V' is constructed starting from some basis vector in E_{j-1} then $\nu_*^{\frac{1}{k_i}} < \nu^{\frac{1}{k_0}}$ due to the ordering of weights and the estimate will be in fact exponentially small.

We conclude that by taking m and n large and so that $m \sim (n^{l_0}\nu^n)^{-\frac{1}{k_0}}$ we can satisfy the condition (1) for any δ while producing the shear bounded below and the size of the images of rectangles bounded above. Then given the desired bound γ on the size of the image we can apply a bounded number of iterates of b to adjust the above estimate for the size to be γ. After that we choose ϵ smaller than the adjusted lower bound for the shear. For this ϵ there exists some δ by the equicontinuity. The above considerations show that we can now take n sufficiently large and $m \sim (n^{l_0}\nu^n)^{-\frac{1}{k_0}}$ to satisfy all three conditions. □

This finishes the proof of Lemma 3.6. □

LEMMA 3.8. *Let W be an invariant subfoliation of W_a^0 and F be an invariant subfoliation of W_a^- for some element a. Suppose that $F \oplus W \subset W_b^-$ for some element b and that the conditional measures of μ on both foliations F and W are atomic. Then the conditional measures on the foliation $F \oplus W$ are also atomic.*

PROOF. We will reduce the lemma to its special case, Lemma 3.9. the reduction goes as follows. Denote $W^I = W \cap W_a^I$. Then Lemma 3.9 shows that the conditional measures on the foliation $F \oplus W^I$ are also atomic.

After that we add root directions of W as in Lemma 3.6 and prove that the conditional measures on $F \oplus W$ are also atomic. The only difference is that in the basic step, Lemma 3.7, we need to control the size of the images of rectangles not only along W^I and previously added root directions but also along F directions. But such control is trivial since by the conditions of the lemma F is contracted by both b and a.

LEMMA 3.9. *Let W be an invariant subfoliation of W_a^I and F be an invariant subfoliation of W_a^- for some element a. Suppose that $F \oplus W \subset W_b^-$ for some element b and that the conditional measures of μ on both foliations F and W are atomic. Then the conditional measures on the foliation $F \oplus W$ are also atomic.*

PROOF. Similar to Lemma 3.6 we prove this lemma inductively by adding one dimensional subfoliations of W to the foliation F one until we exhaust the whole $F \oplus W$. Lemma 3.10 shows that on each step of this process we obtain a foliation with atomic conditional measures.

Let us consider the Jordan normal form of b on the subspace that corresponds to foliation W. This subspace splits over $\mathbb{C}$ into the sum $\bigoplus U_i$ of root subspaces of

b. Each U_i is an eigenspace of a corresponding to some eigenvalue λ_i with $|\lambda_i| = 1$. We denote by ν_i the corresponding eigenvalue of b.

In order to add a one dimensional direction inside some specific U_i for which λ_i or ν_i is complex we replace elements a and b by their proper powers a^{s_i} and b^{t_i} to make the corresponding eigenvalues real: $\lambda_i^{s_i} = 1$ and $0 < \nu_i^{t_i} < 1$. This allows us to consider U_i as a subspace over $\mathbb{R}$.

Unlike Lemma 3.6, for this proof we do not need to follow any particular order of adding. We need only to make sure that when we add a new direction in some U_t we again obtain a subspace which is both a and b invariant and defined over $\mathbb{R}$. The desired partial ordering of directions can be obtained by fixing some Jordan normal basis for the real operator b^{t_i} on U_i.

The next lemma proves the inductive step and completes the proof of Lemma 3.9.

LEMMA 3.10. *Let F_1 and F_2 be invariant subfoliation of W_a^I and F be an invariant subfoliation of W_a^- for some element a. Suppose that F_2 is one-dimensional, $(F \oplus F_1 \oplus F_2) \subset W_b^-$ for some element b, and that the conditional measures of μ on both foliations $F_1 \oplus F_2$ and $F \oplus F_1$ are atomic. Then the conditional measures on the foliation $F \oplus F_1 \oplus F_2$ are also atomic.*

PROOF. Since the conditional measures on the foliations $F_1 \oplus F_2$ and $F \oplus F_1$ are atomic we can find a set of full measure and its compact subset K of measure at least 0.99 which intersect any leaf of the foliations $F_1 \oplus F_2$ and $F \oplus F_1$ in at most one point.

Consider a measurable partition, subordinate to $F \oplus F_1 \oplus F_2$, which consists "mainly" of small "rectangles" of the same size with sides parallel to the foliations F, F_1 and F_2 and has the following property: the measure of the set Int_γ is at least 0.99 for some $\gamma > 0$, where Int_γ consists of points inside rectangles on the distance at least γ from the relative (to the leaf of $F \oplus F_1 \oplus F_2$) boundary of the rectangle that contains the point.

We would like to consider only "good" part of the measure μ, so we introduce a new measure μ_X by $\mu_X(.) = \mu(. \cap X)$, where $X = K \cap Int_\gamma$ with $\mu(X) \geq 0.98$. Let us consider the system of the conditional measures of μ_X w.r.t. the measurable partition into the rectangles (the remaining elements have μ_X measure 0). These measures will be referred to as the conditional measures of rectangles. We regard each rectangle as a direct product of its F, F_1 and F_2 directions. We observe the following dichotomy:

(1) Either for every rectangle, in a set of positive measure, at least $1/3$ of its conditional measure is concentrated on a single $F \oplus F_1$ leaf, hence at one point,
(2) Or there exists a set Y with $\mu_X(Y) > 0.97$ which consists of whole rectangles and a number $d > 0$ such that for any rectangle in Y any subrectangle that carries at least $1/3$ of the conditional measure has width at least d along F_2-direction.

In the first case the existence of atoms for the conditional measures of rectangles implies the existence of atoms for the conditional measures on foliation $F \oplus F_1 \oplus F_2$. As in the proof of Lemma 3.7 we see that since this foliation is contracted by b the existence of atoms forces the conditional measures on $F \oplus F_1 \oplus F_2$ be atomic.

In the latter case each rectangle in Y can be split into three subrectangles of width at least d along one-dimensional F_2 direction so that both the left and the right ones have the conditional measure at least $1/3$. We again may assume that the conditional measures do not have atoms since otherwise we could argue as in the first case.

Now we regard the intersection of the compact set K with any rectangle as a graph of a continuous (not necessarily everywhere defined) function from F-direction to $F_1 \oplus F_2$-direction. The family of these functions is equicontinuous. We would like to show that this equicontinuity contradicts to the recurrence under the action of a properly chosen element.

Since b contracts $F \oplus F_1 \oplus F_2$ we can find sufficiently large number n such that the size of the image of any rectangle under the action of b^n is γ-small. The distance along the F_2 direction between the images of the right and the left subrectangles, which was at least d, becomes at least d'. Let us fix some $\epsilon < d'$ and consider $\delta > 0$ given by the equicontinuity. Since a acts isometrically on $F_1 \oplus F_2$ and contracts F we can choose k so large that the image of any rectangle under $a^k b^n$ is δ-small in F direction. We see that the element $a^k b^n$ satisfies the following conditions:

(1) The image of any rectangle is δ-narrow in the F direction
(2) The distance along the F_2 direction between the images of the right and the left subrectangles is at lest ϵ
(3) The diameter of the image of any rectangle is less than γ, hence the image cannot intersect γ-interiors of two different rectangles simultaneously.

Under these conditions the images of the right and the left subrectangles can not intersect $Z = X \cap Y$ simultaneously. But this implies that $\mu_X(a^k b^n(Z) \cap Z) \leq \frac{2}{3}\mu_X(Z)$ which is impossible since $\mu(Z) = \mu_X(Z)\mu_X(Y) \geq 0.97$. This shows that the second alternative of the dichotomy is impossible and proves that the conditional measures on $F \oplus F_1 \oplus F_2$ are atomic. □

Thus Lemma 3.8 is proved. □

This finished the proof of Theorem 3.1 in the case of atomic conditional measures.

3.5. The case of Haar conditional measures.

LEMMA 3.11. *Let F be an invariant foliation. Suppose that for $\mu - a.e.$ x the conditional measure μ_x^F is a Haar measure on an affine subspace S_x with $\dim S_x = l \geq 1$. Then there exist a finite index subgroup Γ of $\mathbb{Z}^k$, rational subtori $T_i \subset \mathbb{T}^m$, $i = 1, ..., N$, of the same dimension, and Borel probability measures μ_i, $i = 1, ..., N$, on $\mathbb{T}^m$ such that:*

(1) *For $\mu - a.e.$ x the closure of S_x is a translation of T_i for some i,*
(2) *The original measure decomposes as $\mu_{\mathbb{T}^m} = \frac{1}{N}(\mu_1 + ... + \mu_N)$,*
(3) *μ_i, $i = 1, ..., N$ is invariant under the group of translations in the direction of T_i,*
(4) *T_i and μ_i, $i = 1, ..., N$, are invariant under $\alpha(\Gamma)$.*
(5) *All actions $(\alpha(\Gamma), \mu_i)$ are algebraically isomorphic by the toral automorphisms from $\alpha(\mathbb{Z}^k/\Gamma)$.*

PROOF. Let us denote the closure of S_x by $T(x)$. $T(x)$ is a rational subtorus which corresponds to the minimal rational subspace that contains S_x. Hence

$\dim T(x)$ is an invariant function. Since the original measure is ergodic we conclude that $\dim T(x)$ is constant μ-a.e. Let us call two points x and y equivalent if $T(x) = T(y)$. This equivalence relation gives rise to a measurable partition of $\mathbb{T}^m$ into rational subtori.

We first show that the conditional measures on these tori are Lebesgue. Indeed, let us consider a typical torus T with the conditional measure μ_T. The torus T is foliated by affine subspaces S_x in such a way that the conditional measures of μ_T on S_x are Haar for μ_T-a.e. x. One can assign an orthonormal basis to each subspace S_x in a measurable way. This produces a measurable $\mathbb{R}^l$ action on T which preserves μ_T. Let us take the ergodic decomposition of μ_T with respect to this $\mathbb{R}^l$ action and consider one ergodic component. The direction of S_x is invariant under the action since the subspace S_x is the same for all points on one trajectory. Hence almost all subspaces S_x within one ergodic component have the same direction, i.e. are parallel. It follows that the measure on this ergodic component is invariant under translations in this direction. By the construction of T the closure of S_x equals T for all x. This implies that the measure on any ergodic component is a Haar measure on T. Hence μ_T is the Lebesgue measure on T.

Since there can be at most countably many classes of parallel tori we conclude that there exists a torus T_1 such that the set E_1 consisting of all points x for which $T(x)$ is parallel to T_1 has positive measure. By recurrence, for any generator A_j of the action α there exists $n > 0$ such that $\mu_{\mathbb{T}^m}(E_1 \cap A_j^n(E_1)) > 0$. This means that for any point x in this intersection the element A_j^n maps $T(x)$ to a parallel torus. Since A_j^n is an affine map it follows that A_j^n preserves this class of parallel tori. Hence $A_j^n(E_1) = E_1$. In the same way it follows that the set E_1 is invariant with respect to the action of a finite index subgroup $\Gamma \subset \mathbb{Z}^k$. The orbit of the set E_1 consists of finitely many sets E_1, ..., E_N of equal measure which correspond to the elements of $\mathbb{Z}^k/\Gamma$. By ergodicity of the original measure the union of these sets has full measure. Hence $\mu_{\mathbb{T}^m} = \frac{1}{N}(\mu_1 + ... + \mu_N)$, where $\mu_i = \mu_{\mathbb{T}^m}|_{E_i}$. We note that $T(x)$ is parallel to the torus $T_i = a(T_1)$ for μ_i-a.e. x, where a is an element of the action α which maps E_1 to E_i. We conclude that T_i and μ_i are invariant under $\alpha(\Gamma)$ and that μ_i is invariant under the group of translations in the direction of T_i. □

This finishes the proof of Theorem 3.1 □

4. Measure–theoretic rigidity of conjugacies, centralizers, and factors

In this section we apply results about rigidity of invariant measures for actions by toral automorphisms to study fine measurable structure of such actions with respect to *Lebesgue* measure. The conclusions are rather striking: in total contrast to the rank one case (where the automorphisms are Bernoulli and hence the entropy is the only invariant of measurable isomorphism) in the higher rank case fine algebraic information is coded into measurable structure of the action. This section is taken almost *verbatim* from [**KKS**].

4.1. Conjugacies. Suppose α and α' are measurable actions of the same group G by measure–preserving transformations of the spaces (X, μ) and (Y, ν), respectively. If $H : (X, \mu) \to (Y, \nu)$ is a metric isomorphism (conjugacy) between the actions then the lift of the measure μ onto the graph $H \subset X \times Y$ coincides with the lift of ν to graph H^{-1}. The resulting measure η is a very special case of

a *joining* of α and α': it is invariant under the diagonal (product) action $\alpha \times \alpha'$ and its projections to X and Y coincide with μ and ν, respectively. Obviously the projections establish metric isomorphism of the action $\alpha \times \alpha'$ on $(X \times Y, \eta)$ with α on (X, μ) and α' on (Y, ν) correspondingly.

Similarly, if an automorphism $H : (X, \mu) \to (X, \mu)$ commutes with the action α, the lift of μ to graph $H \subset X \times X$ is a self-joining of α, i.e. it is $\alpha \times \alpha$–invariant and both of its projections coincide with μ. Thus an information about invariant measures of the products of different actions as well as the product of an action with itself may give an information about isomorphisms and centralizers.

The use of this joining construction in order to deduce rigidity of isomorphisms and centralizers from properties of invariant measures of the product was first suggested in this context by J.-P. Thouvenot.

In both cases the ergodic properties of the joining would be known because of the isomorphism with the original actions. Very similar considerations apply to the actions of semi–groups by noninvertible measure–preserving transformations. We will use Theorem 3.2

Conclusion of Theorem 3.2 obviously holds for any action of $\mathbb{Z}^k$, $d \geq 2$ which contains a subgroup $\mathbb{Z}^2$ satisfying assumptions of Theorem 3.2. Thus we can deduce the following result which is central for our constructions.

THEOREM 4.1. *Let α and α' be two actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ correspondingly and assume that α satisfies condition $(\mathcal{R})$. Suppose that $H : \mathbb{T}^n \to \mathbb{T}^{n'}$ is a measure–preserving isomorphism between (α, λ) and (α', λ), where λ is Haar measure. Then $n = n'$ and H coincides* (mod 0) *with an affine automorphism on the torus $\mathbb{T}^n$, and hence α and α' are algebraically isomorphic.*

PROOF. First of all, condition $(\mathcal{R})$ is invariant under metric isomorphism, hence α' also satisfies this condition. But ergodicity with respect to Haar measure can also be expressed in terms of the eigenvalues; hence $\alpha \times \alpha'$ also satisfies $(\mathcal{R})$. Now consider the joining measure η on graph $H \subset \mathbb{T}^{n+n'}$. The conditions of Theorem 3.2 are satisfied for the invariant measure η of the action $\alpha \times \alpha'$. Thus η is a translate of Haar measure on a rational $\alpha \times \alpha'$–invariant subtorus $\mathbb{T}' \subset \mathbb{T}^{n+n'} = \mathbb{T}^n \times \mathbb{T}^{n'}$. On the other hand we know that projections of $\mathbb{T}'$ to both $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ preserve Haar measure and are one–to–one. The partitions of $\mathbb{T}'$ into pre–images of points for each of the projections are measurable partitions and Haar measures on elements are conditional measures. This implies that both projections are onto, both partitions are partitions into points, and hence $n = n'$ and $\mathbb{T}' = \text{graph}\, I$, where $I : \mathbb{T}^n \to \mathbb{T}^n$ is an affine automorphism which has to coincide (mod 0) with the measure–preserving isomorphism H. □

We will call two actions α and α' by automorphisms of a $\mathbb{T}^m$ measurably (algebraically) isomorphic up to a time change if for some $C \in GL(m, \mathbb{Z})$ $\alpha \circ C$ is measurably (algebraically) isomorphic to α'. Since a time change is in a sense a trivial modification of an action one is primarily interested in distinguishing actions up to a time change. The corresponding rigidity criterion follows immediately from Theorem 4.1.

COROLLARY 4.1. *Let α and α' be two actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$, respectively, and assume that α satisfies condition $(\mathcal{R})$. If α and α' are measurably isomorphic up to a time change then they are algebraically isomorphic up to a time change.*

4.2. Centralizers. Applying Theorem 4.1 to the case $\alpha = \alpha'$ we immediately obtain rigidity of the centralizers.

COROLLARY 4.2. *Let α be an action of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$. Any invertible Lebesgue measure–preserving transformation commuting with α coincides* (mod 0) *with an affine automorphism of $\mathbb{T}^n$.*

Any affine transformation commuting with α preserves the finite set of fixed points of the action. Hence the centralizer of α in affine automorphisms has a finite index subgroups which consist of automorphisms and which corresponds to the centralizer of $\rho_\alpha(\mathbb{Z}^d)$ in $GL(n,\mathbb{Z})$.

Thus, in contrast with the case of a single automorphism, the centralizer of such an action α is not more than countable, and can be identified with a finite extension of a certain subgroup of $GL(n,\mathbb{Z})$. As an immediate consequence we obtain the following result.

PROPOSITION 4.1. *For any d and k, $2 \le d \le k$, there exists a $\mathbb{Z}^k$–action by hyperbolic toral automorphisms such that its centralizer in the group of Lebesgue measure–preserving transformations is isomorphic to $\{\pm 1\} \times \mathbb{Z}^k$.*

PROOF. Consider a hyperbolic matrix $A \in SL(k+1,\mathbb{Z})$ with irreducible characteristic polynomial and real eigenvalues such that the origin is the only fixed point of F_A. Consider a subgroup of $Z(A)$ isomorphic to $\mathbb{Z}^k$ and containing A as one of its generators. This subgroup determines an embedding $\rho : \mathbb{Z}^d \to SL(k+1,\mathbb{Z})$. It is not difficult to see that all matrices in $\rho(\mathbb{Z}^k)$ are hyperbolic and hence ergodic, condition $(\mathcal{R})$ is satisfied. Hence by Corollary 4.2, the measure–theoretic centralizer of the action α_ρ coincides with its algebraic centralizer, which, in turn, and obviously, coincides with centralizer of the single automorphism F_A isomorphic to $\{\pm 1\} \times \mathbb{Z}^k$. □

4.3. Factors, noninvertible centralizers and weak isomorphism. A small modification of the proof of Theorem 4.1 produces a result about rigidity of factors.

THEOREM 4.2. *Let α and α' be two actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ respectively, and assume that α satisfies condition $(\mathcal{R})$. Suppose that $H : \mathbb{T}^n \to \mathbb{T}^{n'}$ is a Lebesgue measure–preserving transformation such that $H \circ \alpha = \alpha' \circ H$. Then α' also satisfies $(\mathcal{R})$ and H coincides* (mod 0) *with an epimorphism $h : \mathbb{T}^n \to \mathbb{T}^{n'}$ followed by translation. In particular, α' is an algebraic factor of α.*

PROOF. Since α' is a measurable factor of α, every element which is ergodic for α is also ergodic for α'. Hence α' also satisfies condition $(\mathcal{R})$. As before consider the product action $\alpha \times \alpha'$ which now by the same argument also satisfies $(\mathcal{R})$. Take the $\alpha \times \alpha'$ invariant measure $\eta = (\mathrm{Id} \times H)_* \lambda$ on graph H. This measure provides a joining of α and α'. Since $(\alpha \times \alpha', (\mathrm{Id} \times H)_* \lambda)$ is isomorphic to (α, λ) the conditions of Corollary 3.2 are satisfied and η is a translate of Haar measure on an invariant rational subtorus $\mathbb{T}'$. Since $\mathbb{T}'$ projects to the first coordinate one-to-one we deduce that H is an algebraic epimorphism (mod 0) followed by a translation. □

Similarly to the previous subsection the application of Theorem 4.2 to the case $\alpha = \alpha'$ gives a description of the centralizer of α in the group of all measure–preserving transformations.

COROLLARY 4.3. *Let α be an action of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$. Any Lebesgue measure–preserving transformation commuting with α coincides* (mod 0) *with an affine map on $\mathbb{T}^n$.*

THEOREM 4.3. *Let α be an action of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$ and α' another $\mathbb{Z}^d$-action by toral automorphisms. Then (α, λ) is weakly isomorphic to (α', λ') if and only if ρ_α and $\rho_{\alpha'}$ are isomorphic over $\mathbb{Q}$, i.e. if α and α' are finite algebraic factors of each other.*

PROOF. By Theorem 4.2, α and α' are algebraic factors of each other. This implies that α' acts on the torus of the same dimension n and hence both algebraic factor–maps have finite fibers. But this is equivalent to isomorphism over $\mathbb{Q}$ (See [**KKS**], Proposition 2.1). □

Part II. NONUNIFORMLY HYPERBOLIC ACTIONS

The second type of hyperbolicity is *non–uniform*. A discussion of this concept in full generality will appear in [**BKP**, Chapter 1]. A particular, and most important case of non–uniform hyperbolicity appears in the presence of a *hyperbolic measure*, i.e. an invariant measure with non–vanishing Lyapunov exponents (Sections 5 and 7.1).

In the rank one case this is the basis of a very fruitful structural theory at the root of which lie modified Bernoulli and Markov paradigms ([**P, KM, BP, BKP**]). Not surprisingly, the analogy with the uniform case extends to the actions of higher–rank groups and hyperbolic measures display certain rigidity properties not dissimilar to those for Anosov actions. In Sections 5, 6, and 7 we describe some of the principal technical tools and certain results concerning the nonuniformly hyperbolic situation. Some of these tools can be naturally defined in greater generality than the case of actions by diffeomorphisms of compact manifolds which is of principal interests for us and we develop appropriate contexts accordingly. In the last section we indicate how certain rigidity results appear in this context. The two most important ingredients of this approach is the concept of Lyapunov Hölder structure which is the proper framework for the cocycle rigidity results and the generalization of approach from Section 2.2 based on the consideration of conditional measures on contracting foliations. Detailed presentation of these results will appear in a separate paper.

5. Linear extensions of $\mathbb{Z}^k$ actions

We begin with a general treatment of Lyapunov characteristic exponents which can be defined for the linear extensions of actions by measure–preserving transformations of a finite measure space. To simplify notations a bit we will restrict the discussion to the discrete time case ($\mathbb{Z}^k$ actions); the results for the $\mathbb{R}^k$ case can actually be deduced from those for $\mathbb{Z}^k$ by taking the $\mathbb{Z}^k$ subgroup in an $\mathbb{R}^k$ action and showing that its Lyapunov decomposition is if fact $\mathbb{R}^k$ invariant.

5.1. Lyapunov exponents and Lyapunov metric. Let (X, μ) be a probability Lebesgue space $\alpha : \mathbb{Z}^k \times X \to X$ an ergodic action of the group $\mathbb{Z}^k$ by measure–preserving transformations of the space (X, μ), $A : \mathbb{Z}^k \times X \times \mathbb{R}^m \to X \times \mathbb{R}^m$

a *linear extension* of the action α. Such an extension is determined by a *matrix–valued cocycle* $\mathcal{A} : \mathbb{Z}^k \times X \to GL(m, \mathbb{R})$ as follows

$$A(n, x, t) = (\alpha(n, x), \mathcal{A}(n, x)(t)),$$

where by the group property

$$\mathcal{A}(n_1 + n_2, x) = \mathcal{A}(n_2, \alpha(n_1, x))\mathcal{A}(n_1, x). \tag{5.1.1}$$

THEOREM 5.1 (Multiplicative Ergodic Theorem for $\mathbb{Z}^k$ actions). *Suppose for each* $n \in \mathbb{Z}^k$,

$$\log \|\mathcal{A}(n, x)\| \in L^1(X, \mu). \tag{5.1.2}$$

Then there exist linear functionals $\chi_1, \ldots, \chi_l$ *on* $\mathbb{R}^k$ *and for* μ*-a.e. point* $x \in X$ *a decomposition of the fiber* $\mathbb{R}^m_x$ *over* x*:*

$$\mathbb{R}^m_x = E_{\chi_1}(x) \oplus \cdots \oplus E_{\chi_l}(x) \tag{5.1.3}$$

such that for $i = 1, \ldots, l$ *and for any* $v \in E_{\chi_i}(x)$ *one has*

$$\lim_{n \to \infty} \frac{\log \|\mathcal{A}(n, x)t)\| - \chi_i(n)}{\|n\|} = 0. \tag{5.1.4}$$

and

$$\lim_{n \to \infty} \frac{\log \det \mathcal{A}(n, x) - \sum_{i=1}^{l} m_i \chi_i(n)}{\|n\|} = 0, \tag{5.1.5}$$

where $m_i = \dim E_{\chi_i}(x)$*. Moreover, the set of points where this decomposition is defined is* α*–invariant and decomposition itself is* A*–invariant.*

Definitions. The functionals $\chi_1, \ldots, \chi_l$ are called the *Lyapunov characteristic exponents* of A. The dimension m_i of the space $E_{\chi_i}(x)$ is called the *multiplicity* of the exponent χ_i. The decomposition (5.1.3) is called the *(fine) Lyapunov decomposition* at the point x. Points where the assertions of the theorem are satisfied will be called *regular*. The set of all regular points will be usually denoted by the letter Λ, sometimes with extra indices. A basis $t_1, \ldots, t_m$ of $\mathbb{R}^m_x$ for a regular point x is called *regular* if it agrees with the Lyapunov decomposition (5.1.3) i.e. can be split into bases of the spaces $E_{\chi_1}(x), \ldots, E_{\chi_l}(x)$.

Remarks.

1. Due to the cocycle equation (5.1.1) it is sufficient to require the integrability condition (5.1.2) only for a subset of elements which generate $\mathbb{Z}^k$ as a semigroup, e.g. for the standard generators and their opposites.
2. For any $n \in \mathbb{Z}^k$ the values $\chi_i(n), i = 1, \ldots, l$ are equal to the Lyapunov characteristic exponent of the extension $A(n)$ of the map $\alpha(n)$ in the usual sense [**KM**],Section S2. Even though the map $\alpha(n)$ may not be ergodic with respect to μ its Lyapunov exponents are invariant with respect to the action of the whole group $\mathbb{Z}^k$ and hence constant almost everywhere. Naturally, for a particular n the values of $\chi_i(n)$ may coincide for some i so the map $F(n)$ may have fewer than l distinct exponents with higher multiplicities.
3. One may think of the group $\mathbb{Z}^k$ as embedded into $\mathbb{R}^k$ as the standard integer lattice. The values $\chi_i(t)$ for $t \in \mathbb{R}^k \setminus \mathbb{Z}^k$ do not make sense in the context of the action α. However they are easily interpreted as the Lyapunov characteristic exponents of the elements of the *suspension* action.

Theorem 5.1 can be easily deduced from the standard Oseledets Multiplicative ergodic theorem for linear extensions of a single measure–preserving transformation [**O**],[**BKP**] by a simple induction process. Namely, one first applies Oseledets theorem to the first generator of the $\mathbb{Z}^k$ action. The Lyapunov decomposition for it is invariant under the whole action so one can apply Oseledets theorem to the restriction of the extension of the second generator to each element of the Lyapunov decomposition and so on. Once one obtained a decomposition for which limits exist for multiples of all generators, existence of the limits (5.1.4), (5.1.5) and the linearity of exponents follow easily from the cocycle relation (5.1.1).

Denote the standard scalar product in $\mathbb{R}^m$ by $< \cdot, \cdot >$. Given $\epsilon > 0$ and a regular point x we will call *the standard ϵ–Lyapunov scalar product (or metric)* the scalar product $< \cdot, \cdot >_{x,\epsilon}$ in $\mathbb{R}^m_x$ defined as follows:

For $u, v \in E_{\chi_i}(x),\ i = 1, \dots, l$:

$$< u, v >_{x,\epsilon} = \sum_{n \in \mathbb{Z}^k} < \mathcal{A}(n,x)u, \mathcal{A}(n,x)v > \exp(-2\chi_i(n) - \epsilon\|n\|),$$

where the series converges due to (5.1.4).

For $u \in E_{\chi_i}(x),\ v \in E_{\chi_j}(x),\ i \neq j \quad < u, v >_{x,\epsilon} = 0$.

We will usually omit the word "standard" and will call this scalar product *ϵ–Lyapunov scalar product*, or, if ϵ has been fixed and no confusion may appear, simply *Lyapunov scalar product*. The norm generated by this scalar product will be called the (standard ϵ–) Lyapunov norm and denoted by $\| \cdot \|_{x,\epsilon}$

By construction of the standard ϵ–Lyapunov norm one has for $u \in E_{\chi_i}(x)$,

$$\exp(\chi_i(n) - \epsilon\|n\|)\|u\|_{x,\epsilon} \leq \|\mathcal{A}(n,x)u)\|_{x,\epsilon} \leq \exp(\chi_i(n) + \epsilon\|n\|)\|u\|_{x,\epsilon}. \tag{5.1.6}$$

Notice that by definition $\|u\|_{x,\epsilon} \geq m^{-1/2}\|u\|$.

Denote for a $\delta > 0$ by $\Lambda_{\epsilon,\delta}$ the following subset of the set of regular points:

$$\Lambda_{\epsilon,\delta} = \{x \in X :\ \forall u \in \{x\} \oplus \mathbb{R}^m,\ \|u\|_{x,\epsilon} \leq \delta^{-1}\|u\|\}. \tag{5.1.7}$$

A real–valued function ϕ on X is called *tempered at x* (with respect to the action α) if

$$\lim_{n \to \infty} \frac{\phi(\alpha(n,x))}{n} = 0.$$

It is easy to see that a tempered function can be estimated from above by a *slowly changing* function. Namely, let ϕ be tempered and let $\epsilon > 0$. We define

$$\phi_\epsilon(x) = \max_{n \in \mathbb{Z}^k}(|\phi(\alpha(n,x))| - \epsilon\|n\|).$$

Obviously $\phi_\epsilon(x) \geq |\phi(x)|$. A simple calculation shows that ϕ_ϵ indeed changes slowly:

$$\phi_\epsilon(x) - \epsilon\|n\| \leq \phi_\epsilon(\alpha(n,x)) \leq \phi_\epsilon(x) + \epsilon\|n\|. \tag{5.1.8}$$

The following fact is proved in the same way as for $k = 1$ in [**P**],[**BKP**].

PROPOSITION 5.1 (Temperedness of the Lyapunov metrics). *For any regular point $x \in X$ any non-zero $u \in \mathbb{R}^m_x$ and any $\epsilon > 0$ the function $\log \frac{\|u\|_{x,\epsilon}}{\|u\|}$ is tempered.*

Two cocycles $\mathcal{A}$ and $\mathcal{B}$ over the same action α are called *equivalent* if there exists a function $C : X \to GL(m, \mathbb{R})$ such that

$$\max\{\log \|C\|, \log \|C^{-1}\|\}$$

is a tempered function and

$$\mathcal{B}(n,x) = C^{-1}(x)\mathcal{A}(n,x)C(\alpha(x,n)).$$

Equivalent cocycles have the same Lyapunov characteristic exponents with the same multiplicities. The above proposition implies that any measurable coordinate change which takes the standard scalar product to the standard ϵ–Lyapunov scalar product is tempered. Thus any cocycle $\mathcal{A}$ satisfying (5.1.2) is equivalent to a block cocycle with $m_i \times m_i$ diagonal blocks $\mathcal{A}_i(n,x)$, $i = 1,\dots,l$ such that

$$\|\mathcal{A}_i(n,x)\| \le \exp(\chi_i(n) + \epsilon\|n\|) \quad \text{and} \quad \|\mathcal{A}_i^{-1}(n,x)\| \ge \exp(-\chi_i(n) - \epsilon\|n\|).$$

We will call any measurable coordinate change of this kind an *ϵ–Lyapunov coordinate change* an the resulting equivalent cocycle an *ϵ–reduced* cocycle.

5.2. Lyapunov hyperplanes and Weyl chambers. Now we can generalize certain notions previously introduced in Sections 1.2.3 and 1.3 for special cases.

The hyperplane $\ker\chi \subset \mathbb{R}^k$, where χ is a non-zero Lyapunov exponent, is called a *Lyapunov hyperplane.* The subspace $\chi^{-1}(-\infty,0)$ (corr $\chi^{-1}(0,\infty)$) is called a *negative* (corr. *positive) Lyapunov half-space.* A Lyapunov hyperplane L is called *rational* if $L \cap \mathbb{Z}^k$ is a lattice in L, *totally irrational* if $L \cap \mathbb{Z}^k = \{0\}$, and *partially irrational* otherwise. For $k = 2$ we will call Lyapunov hyperplanes *Lyapunov lines.* Naturally , every Lyapunov line is either rational or totally irrational; in the latter case we will call it simply irrational.

A linear extension of an ergodic $\mathbb{Z}^k$ action is called *partially hyperbolic* if there is at least one non-zero Lyapunov exponent and *hyperbolic* if all Lyapunov exponents are different from zero.

A hyperbolic linear extension is called *totally non–symplectic (TNS)* if non of the Lyapunov exponents is proportional to another with the negative coefficient of proportionality.

An element $n \in \mathbb{Z}^k$ is called *regular* if n does not belong to any of the Lyapunov hyperplanes. A regular element for a hyperbolic linear extension of a $\mathbb{Z}^k$ action is called *hyperbolic.* A *Weyl chamber* is a connected component of the complement to the union of all Lyapunov hyperplanes.

Each Weyl chamber is an open convex polyhedral cone in $\mathbb{R}^k$. Inside a Weyl chamber every non-zero Lyapunov exponent has a constant sign. Conversely, the locus of points in $\mathbb{R}^k$ for which each non-zero Lyapunov exponent has a particular sign is either empty or is a Weyl chamber. Thus any Weyl chamber can be characterized as a minimal non-empty intersection of positive and negative Lyapunov half-spaces.

For a partially hyperbolic element $n \in \mathbb{Z}^k$ and a regular point x we set

$$E_n^+(x) = \bigoplus_{i:\chi_i(n)>0} E_{\chi_i}(x) \text{ and } E_n^-(x) = \bigoplus_{i:\chi_i(n)<0} E_{\chi_i}(x).$$

These subspaces are called correspondingly the *expanding* (or *unstable*) and the *contracting* (or *stable*) subspaces for n at the point x. Notice that for any $n \in \mathbb{Z}^k$ if one takes in the definition of the standard ϵ–Lyapunov metric $\epsilon < \epsilon_0(n) = \frac{\min|\chi_i(n)|}{\|n\|}$ where the minimum is taken over all exponents which do not vanish at n, then the extension $A(n)$ uniformly contracts the ϵ–standard Lyapunov metric in the contracting subspaces of all regular points and uniformly expands it in the expanding subspaces of those points. By making an ϵ–Lyapunov coordinate change

we obtain an equivalent cocycle generating an extension B such that $B(n)$ uniformly contracts the standard Euclidean metric in the contracting subspaces of all regular points and uniformly expands it in the expanding subspaces of those points.

By definition contracting and expanding subspaces are the same for all elements in the same Weyl chamber. Any minimal non-zero intersection of expanding subspaces for various regular elements corresponds to a positive Lyapunov half-space H and will be denoted by $E_H(x)$. It is equal to the sum of the subspaces E_{χ_i} for those χ_i's which are positive on the subspace H. In other words, all these exponents are proportional with positive proportionality coefficients. Thus for any Lyapunov half–space H one can find a uniquely defined Lyapunov characteristic exponent $\chi(H)$ (called the *bottom exponent* for H) and positive numbers $1 = c_1(H) < c_2(H) <, \dots, < c_{m(H)}(H)$ such that

$$E_H(x) = \bigoplus_{i=1}^{m(H)} E_{c_i(H)\chi(H)}.$$

Let us denote

$$E^i_H(x) = \bigoplus_{j=i}^{m(H)} E_{c_j(H)\chi(H)}.$$

The nested sequence $E_H(x) = E^1_H(x) \supset E^2_H(x) \supset \cdots \supset E^{m(H)}_H(x) = E_{c_{m(H)}(H)\chi(H)}$ is called the *upper filtration* of the space $E_H(x)$.

Number all positive Lyapunov half–spaces by $H_1, \dots, H_s$. We will call the decomposition

$$\mathbb{R}^m_x = \bigoplus_{i=1}^{s} E_{H_i}(x)$$

the *coarse Lyapunov decomposition* at the point x. Extension A is called *multiplicity–free* if all Lyapunov exponents are simple and no two of them are positively proportional. In other words, in the multiplicity–free case the elements of the coarse Lyapunov decomposition are one–dimensional.

5.3. Strongly hyperbolic extensions. In the case of multiple exponents it is convenient to count each exponent the number of times equal to its multiplicity so that there are always exactly m exponents. Since this will not cause any confusion, from now on we will denote exponents listed this way by $\chi_1, \dots, \chi_m$, unless explicitly stated otherwise. This allows us to define the *Lyapunov map* $\Psi_A : \mathbb{R}^k \to \mathbb{R}^m$ by $\Psi_A = (\chi_1, \dots, \chi_m)$. The Lyapunov map is defined up to a permutation of coordinates.

Thus the linear extension μ is hyperbolic if and only if $\operatorname{Im}\Psi_A$ does not lie in any coordinate hyperplane. We will call $\dim \operatorname{Ker}\Psi_A$ the *defect* of μ and denote it by $d(A)$. Equivalently, $d(A)$ is equal to the dimension of the intersection of all Lyapunov hyperplanes. Furthermore, $\dim \operatorname{Im}\Psi_A$ is called the *rank* of A and is denoted by $r(A)$. Equivalently, $r(A)$ is equal to the maximal number of linearly independent Lyapunov exponents.

Definition A hyperbolic linear extension A is called *strongly hyperbolic* if $d(A) = 0$, i.e. if the intersection of all Lyapunov hyperplanes consists of the origin.

Obviously, $d(A) + r(A) = k$ and since $r(A) \leq m$

$$k \leq m + d(A).$$

In particular, for any strongly hyperbolic measure

$$k \leq m. \tag{5.3.1}$$

5.4. Resonances. Let $\chi_1, \ldots, \chi_l$ be the different Lyapunov exponents of an extension A as in Section 1.1.We will say that A possesses an *essential resonance* if the following two conditions hold:

($\mathcal{R}1$) There exist $i \in \{1, \ldots, l\}$ and non–negative integers $s_1, \ldots, s_l$ such that

$$\chi_i = \sum_{j \neq i} s_j \chi_j \tag{5.4.1}$$

($\mathcal{R}2$) There exists a Weyl chamber $\mathcal{W}$ such that in $\mathcal{W}$ $\chi_i < 0$ and $\chi_j < 0$ for all j such that $s_j \neq 0$.

If A does not possess any essential resonances we will call it *essentially non-resonant.* If in addition all exponents are simple (multiplicity one) we will say that A is *non–resonant.*

Let $x_1, \ldots, x_l$ be negative numbers. We will call a relation of the form

$$x_i = \sum_{j \neq i} s_j x_j, \tag{5.4.2}$$

where $s_1, \ldots, s_l$ are non–negative integers, a *resonance relation* between $x_1, \ldots, x_l$. Notice that any given set of negative numbers may satisfy only finitely many resonance relations.

PROPOSITION 5.2. *There exists an element $n \in \mathbb{Z}^k \cap \mathcal{W}$ such that*

(1) *The multiplicities of the Lyapunov exponents for the extension $A(n)$ are the same as for the extension A.*
(2) *Any resonance relation between the negative Lyapunov characteristic exponents of $A(n)$ comes from an essential resonance for A.*

A proof of this statement is contained in the proof of [**K1**], Theorem 4.4.

6. Normal forms and linearization of non-linear extensions

6.1. Extensions by non-linear contractions. We will be interested later on in the way a smooth $\mathbb{Z}^k$ action with a partially hyperbolic invariant measure acts on the family of stable manifolds of its regular element. This naturally leads us to the study of measurable extensions of a measure–preserving transformation by diffeomorphisms preserving the zero section with negative Lyapunov exponents for the derivative at that section, as well as centralizers of such extensions.

Let as before (X, μ) be a probability Lebesgue space and $\alpha : \mathbb{Z}^k \times X \to X$ be an ergodic action of the group $\mathbb{Z}^k$ by measure–preserving transformations of the space (X, μ). We will assume that an extension Φ of α is defined in a neighborhood of the zero section, preserves the zero section and acts by C^∞ diffeomorphisms in the fibers. In other words, Φ can be written in coordinates $(x, t) \in X \times \mathbb{R}^m$ as

$$\Phi(n, x, t) = (\alpha(n, x), \mathcal{F}(n, x)(t)), \tag{6.1.1}$$

where $\mathcal{F}(n, 0) = 0$ and $\mathcal{F}$ is C^∞ in t.

The derivatives in the t variable at the zero section which we will denote simply by $D\mathcal{F}_0(n, x)$ define a linear extension of α. We assume that for some $n_0 \in \mathbb{Z}^k$, $D\mathcal{F}_0(n_0, x)$ has negative characteristic exponents. We will also assume that all partial derivatives of all orders are tempered functions.

We will consider the following natural notion of equivalence for extensions of the form (6.1.1). Namely, we will allow coordinate changes which are measurable in x, preserve each fiber $\mathbb{R}^m_x$, fix the origin, are C^∞ in each fiber and have tempered logarithm of all derivatives of all orders at the zero section. We will call such coordinate changes *admissible*. Our main purpose is to show that any extension can be brought to a certain normal form by an admissible coordinate change. We will do it by first considering the element $F(n_0)$. We will find a normal form for it and them describe the centralizer of that normal form. This description will show that the admissible coordinate change bringing $F(n_0)$ to its normal form also brings the whole extension F to a certain normal form. The simplest kind of such a normal form is linear. The obstacles to linearization of $F(n_0)$ via an admissible coordinate change are resonance relations between the Lyapunov characteristic exponents of the derivative at the zero section. Presence of such relations might force existence of certain non–linear resonance terms in the normal form.

6.2. Normal forms. The results of this sub–section represent one of several natural "non–stationary" generalizations of well-known facts about normal forms of smooth contractions near a fixed point or an invariant manifold which can be for example in [**Be**] and [**BK**]. Another version of the non–stationary normal form theory for contractions adapted to the study of uniformly hyperbolic dynamical systems is developed in [**GK**] and [**G**]. The latter version is a crucial in the proofs of local differentiable rigidity both for actions of abelian groups [**KS2**], as well as higher rank Lie groups and their lattices [**MQ**].

Let (X, μ) be a probability Lebesgue space, $f : X \to X$ be a measure–preserving transformation of (X, μ), not necessarily ergodic, U an open neighborhood of the origin in $\mathbb{R}^m$ and $\Phi : X \times U \to X \times \mathbb{R}^m$ an extension of f satisfying assumptions corresponding those for $\Phi(n_0)$ above. Namely, it preserves the zero section, is C^∞ along the fibers, the Lyapunov characteristic exponents for the linear extension defined by the derivatives at zero section are constant almost everywhere and negative and all higher derivatives are tempered functions. Let $\chi_i, \dots \chi_l$ be different Lyapunov characteristic exponents of the derivative extension and $m_1, \dots, m_l$ be their multiplicities. Consider all the resonance relations (5.4.2) between the numbers $\chi_i, \dots \chi_l$. Represent $\mathbb{R}^m$ as the direct sum of the spaces $\mathbb{R}^{m_i}, \dots, \mathbb{R}^{m_l}$ and let $(t_1, \dots, t_l)$ be the corresponding coordinate representation of a vector $t \in \mathbb{R}^m$. Let $P : \mathbb{R}^m \to \mathbb{R}^m$; $(t_1, \dots, t_l) \mapsto (P_1(t_1, \dots, t_l), \dots, P_l(t_1, \dots, t_l))$ be a polynomial map preserving the origin. We will say that the map P is of *resonance type* if it contains only such homogeneous terms in $P_i(t_1, \dots, t_l)$ with degree of homogeneity s_j in the coordinates of t_j, $i = 1, \dots, l$ for which the resonance relation $\chi_i = \sum_{j \neq i} s_j \chi_j$ holds. It is easy to see that polynomial maps of the resonance type with invertible derivative at the origin form a group. Since there are only finitely many resonance relations between $\chi_1, \dots, \chi_l$ this is a finite–dimensional Lie group. We will denote this group by G_χ. In particular, if there are no resonance relations between the numbers $\chi_1, \dots \chi_l$ then $G_\chi = GL(m, \mathbb{R})$, the group of linear automorphisms of $\mathbb{R}^m$.

THEOREM 6.1 (non–stationary normal form for contractions). *There exists an admissible coordinate change in $X \times \mathbb{R}^m$ which transforms Φ to an extension Ψ of the following form*

$$\Psi(x, t) = (f(x), \mathcal{P}_x(t)) \tag{6.2.1}$$

where for almost every $x \in X$, $\mathcal{P}_x \in G_\chi$.

The proof of Theorem 6.1 follows one of the usual schemes in the normal form theory. (see, e.g. [**KH**, Section 6.6], for the local case and [**GK**] for the uniformly hyperbolic nonstationary case). It includes three steps:

Step 1 Finding a tempered *formal* coordinate change, i.e. the Taylor series at the zero section for the desired coordinate change.

Step 2 Constructing a tempered smooth (C^∞) coordinate change for which the formal power series found at Step 1 is the Taylor series at the zero section

Step 3 The coordinate change constructed at Step 2 conjugates our extension with an extension which is C^∞ *tangent* to the derivative extension. We show that any two tempered C^∞ tangent contracting extensions are conjugate via a C^∞ tempered coordinate change C^∞ tangent to identity

We will outline Step 1 since at that step the role played by resonances becomes apparent. In particular it becomes clear that the nonuniform case is a more direct generalization of the local one and the Lyapunov exponents play exactly the same role as the eigenvalues. In the uniform case considered in [**GK**] an extra narrow band condition is required and not only resonance but also subresonance terms have to be allowed in the normal form. Of course the price paid for this simplification is that the dependence on the base point in the nonuniform nonstationary normal form is only measurable even if all the data are smooth.

Steps 2 and 3 are very similar to the uniform nonstationary arguments in [**GK**] since the use of Lyapunov metrics essentially reduces our situation to the case of an extension by uniform contractions.

First we make a linear coordinate which brings the derivative at the zero section to the block form as described at the end of Section 5.1. Denote the $m_i \times m_i$, $i = 1, \dots, l$ blocks thus obtained by $\Phi_1(x), \dots, \Phi_l(x)$. Denote the coordinates corresponding this block form by $t_1, \dots, t_l$, where $t_i \in \mathbb{R}^{m_i}$.

We will look inductively for successive jets of the desired coordinate change at the zero section to eliminate the non–resonance terms of higher and higher order.The base of the induction is the linear coordinate change described above. At the nth inductive step we start form the extension whose nth jets are polynomials of resonance type. We will find a coordinate change with the identity linear part and polynomial homogeneous non–linear part of degree $n+1$. Composing this coordinate change with the result from the previous steps would not change the lower order terms. The t_i component $C_x(t_1, \dots, t_l)$ of the coordinate change with degree of homogeneity s_j in the coordinates of t_j, $j = 1 \dots, l$ satisfies the following functional equation

$$\Phi_i(x) C_x(t_1, \dots, t_l) - C_{f(x)}(\Phi_1(x)t_1, \dots, \Phi_l(x)t_l) = H_x(t),$$

where $H_x(t)$ is a known function determined by the previous steps of the process. Let for $t = (t_1, \dots, t_l)$, $\Phi(t) = (\Phi_1(t_1), \dots, \Phi_l(t_l))$. In the non–resonance case we can solve this equation by one of the two "telescoping sums": if

$$\chi_i > \sum s_j \chi_j, \tag{6.2.2}$$

then

(6.2.3)

$$C_x(t) = \sum_{m=0}^{\infty} \Phi_i^{-1}(x) \dots \Phi_i^{-1}(f^m(x)) H_{f^m(x)}(\Phi(f^m(x)) \dots \Phi(f(x))(\Phi(x))(t)),$$

where the series in the right–hand part converges exponentially; if

$$\chi_i < \sum s_j \chi_j, \tag{6.2.4}$$

then

$$C_x(t) = \tag{6.2.5}$$
$$-\sum_{m=1}^{\infty} \Phi_i(f^{-1}(x)) \dots \Phi_i(f^{-m}(x)) H_{f^{-m}(x)}(\Phi^{-1}(f^{-m}(x)) \dots \Phi^{-1}(f^{-1}(x))(t)),$$

where again the series converges exponentially. Notice that the former case holds for all but finitely many homogeneous terms. The resonance terms of degree $n+1$ do not change, at this and successive steps. However that they were likely to have changed at the previous steps. Thus, while the type of normal form is determined by the first jet (the derivative extension), coefficients of its nth jet depend on the $(n-1)$st jet of the original extension.

THEOREM 6.2 (Centralizer for resonance maps). *Let g be a non–singular (not necessarily measure–preserving) transformation of the space (X, μ) commuting with f, and $\Gamma(x,t) = (g(x), \mathcal{G}_x(t))$ be an extension of g by C^∞ local diffeomorphisms preserving the zero section and commuting with an extension Ψ of the form (6.2.1). Then for almost every $x \in X$, $\mathcal{G}_x \in G_\chi$.*

For certain applications it is useful to define another class of polynomial maps associated with a given set of Lyapunov exponents. As before let $P : \mathbb{R}^m \to \mathbb{R}^m$; $(t_1, \dots, t_l) \mapsto (P_1(t_1, \dots, t_l), \dots, P_l(t_1, \dots, t_l))$ be a polynomial map preserving the origin. We will say that the map P is of *sub–resonance type* if it has homogeneous terms in $P_i(t_1, \dots, t_l)$ with degree of homogeneity s_j in the coordinates of t_j, $i = 1, \dots, l$ only if

$$\chi_i \le \sum_{j \ne i} s_j \chi_j.$$

In other words, in addition to resonance terms we also allow terms satisfying the strict inequality (6.2.4). The product of two maps of sub–resonance type is again a map of sub–resonance type; thus, as in the case of resonance maps, polynomial maps of sub–resonance type with invertible derivative at the origin form a group. It is larger than G_χ but still finite dimensional. We will denote this group by SR_χ. In particular, if $\min \chi_i > 2 \max \chi_i$, then $SR_\chi = GL(m, \mathbb{R})$. Theorem 6.2 has a counterpart for maps of sub–resonance type.

THEOREM 6.3 (Centralizer for sub–resonance maps). *Let Ψ be an extension of a measure–preserving transformation f; $(X, \mu \to (X, \mu)$ of the following form*

$$\Psi(x,t) = (f(x), \mathcal{P}_x(t)), \text{ where for almost every } x \in X, \ \mathcal{P}_x \in SR_\chi.$$

Let g be a non–singular transformation of the space (X, μ) commuting with f, and $\Gamma(x,t) = (g(x), \mathcal{G}_x(t))$ be an extension of g by C^∞ local diffeomorphisms preserving the zero section and commuting with an extension Ψ . Then for almost every $x \in X$, $\mathcal{G}_x \in SR_\chi$.

The proofs of theorems 6.2 and 6.3 are similar to the proof of Theorem 1.2 from [**GK**] with a proper allowance for non–uniformity of the situation. Details will appear in a separate paper.

7. Pesin theory for $\mathbb{Z}^k$ actions

7.1. Local $\mathbb{Z}^k$ actions and rank restrictions.

7.1.1. *Locally maximal sets.* The main situation when the general setup described in the previous two sections arises is the extension of a smooth $\mathbb{Z}^k$ action by derivatives. A somewhat more general semi-local setup is as follows.

Let M be a differentiable manifold, $U \subset M$ an open set and $\Lambda \subset U$ a compact subset. Let $f_1, \dots, f_k : U \to M$ be commuting diffeomorphic embeddings of class $C^{1+\epsilon}$ for some $\epsilon > 0$ preserving the set Λ. Obviously, the maps $f_1, \dots, f_k$ restricted to the set Λ generate an action of the group $\mathbb{Z}^k$ on Λ. Outside Λ the action of the whole group $\mathbb{Z}^k$ may not be defined. We will call this situation a *local* $\mathbb{Z}^k$ *action near* Λ. Our standard notation for a local action will be F so that for $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$ one has $F(n_1, \dots, n_k) = f_1^{n_1} \circ \cdots \circ f_k^{n_k}$. The set Λ is called a *locally maximal set* for the local action F if for some open set $V \supset \Lambda$ the set Λ is the biggest invariant set contained in V, i.e.

$$\Lambda = \bigcap_{n \in \mathbb{Z}^k} F(n)V.$$

Any such set V will be called a *separating neighborhood* for Λ.

7.1.2. *Lyapunov exponents and hyperbolic measures for local actions.* Let μ be a Borel probability F-invariant ergodic measure such that $\operatorname{supp} \mu \subset \Lambda$. Even though the tangent bundle TM or its restriction to Λ, $T_\Lambda M$ may not be trivial, it is always trivial up to a set of measure zero. So the derivatives of the local action DF acting on $T_\Lambda M$ generate a linear extension of F as described in Section 5. Different trivializations define cohomologous linear extensions (see [**KM**]) so various properties of this extensions discussed in Section 5 do not depend on a particular choice of the trivialization. Accordingly we will attribute these properties to the action F itself. Thus we will speak of the Lyapunov characteristic exponents, hyperbolic measures, Weyl chambers, etc of a local action.

In the smooth situation various feature of the structure of the derivative extension are more of less directly reflected in the structure of the (non-linear) action itself. Accordingly we will associate the attributes of the derivative extension to the action and the measure. Notice that cocycles associated to the derivative extension depend on a choice of a measurable trivialization of the tangent bundle but cocycles corresponding to different trivializations are cohomologous. Thus Lyapunov exponents for the derivative cocycle with respect to an ergodic invariant measure are constant almost everywhere and thus are well defined as well as all notions derived form those exponents: Lyapunov hyperplanes, Weyl chambers, Lyapunov map, rank, defect, etc. We will thus speak about Lyapunov exponents of a local action with respect to an invariant measure, about partially hyperbolic, strongly hyperbolic and TNS (totally nonsymplectic) measures. In this case we will use the notation Ψ_μ for the Lyapunov map, $r(\mu)$ for the rank and so on.

7.1.3. *TNS actions.* Cartan actions ([**KKS**]) of which the $\mathbb{Z}^2$ actions of section 2.1 are the simplest examples are TNS. See [**NT**, Section 5] for a detailed discussion of TNS actions, in particular for specific examples of such actions on Tori and nilmanifolds. Among algebraic Anosov actions discussed in Section 1.3 TNS actions appear only among actions by automorphisms of tori and nilmanifolds, their finite affine extensions and suspensions. Most of the other interesting examples, are related in one way or another with semisimple Lie groups, and hence possess at

least a partial simplectic structure in certain transverse directions. Hence some Lyapunov exponents appear in pairs which differ by the sign. See the description of standard Anosov actions in [**KS1**].

7.1.4. *The rank restriction.*

PROPOSITION 7.1 ([**K1**]). *If* $\operatorname{Im}\Psi_\mu$ *intersects the positive octant* $\mathbb{R}^n_+$ *(and hence also the negative octant* $-\mathbb{R}^n_+$*) then the measure* μ *is atomic.*

COROLLARY 7.1. *If* μ *is a hyperbolic measure for a local* $\mathbb{Z}^k$ *action and* $k = \dim M + d(\mu)$*, then* μ *is atomic. In particular, any strongly hyperbolic ergodic invariant measure for a local* $\mathbb{Z}^k$ *action on a* k*–dimensional manifold is atomic.*

PROOF. Obviously $r(\mu) = \dim M$ so $\operatorname{Im}\Psi_\mu = \mathbb{R}^{\dim M} \supset \mathbb{R}^{\dim M}_+$ □

Corollary 7.1 immediately implies that if a local $\mathbb{Z}^k$ action on an n–dimensional manifold possesses a strongly hyperbolic non–atomic invariant measure than

$$k \leq n - 1. \tag{7.1.1}$$

This inequality cannot be improved. In fact, for any $n \geq 2$ there is a $\mathbb{Z}^{n-1}$ action on the n–dimensional torus $\mathbb{T}^n$ by hyperbolic automorphisms for which Lebesgue measure is invariant and strongly hyperbolic. A "surgery" described in [**KL**] allows to produce actions of $\mathbb{Z}^{n-1}$ with strongly hyperbolic absolutely continuous invariant measures on certain n–dimensional manifolds other than torus. The rank of these measures is equal to $n - 1$.

A certain feature of these examples holds in general.

COROLLARY 7.2. *If* μ *is a non-atomic hyperbolic measure for a local* $\mathbb{Z}^k$ *action* F *and* $r(\mu) = \dim M - 1$ *then there exists a regular element* $n \in \mathbb{Z}^k$ *such that* $F(n)$ *has exactly one positive Lyapunov exponent and this exponent is simple.*

Further results of [**K1**] show that in the low–dimensional situations, *i.e.* for $n = 2$ or 3 the only way for a non–atomic hyperbolic measure to violate (7.1.1) is to have elements in the $\mathbb{Z}^k$ action with massive (in particular, full measure) sets of fixed points. In the real–analytic case this leads to the following conclusion:

PROPOSITION 7.2 ([**K1**]). *Any hyperbolic invariant measure for an effective real– analytic action of* $\mathbb{Z}^2$ *on a compact (real–analytic) surface or an effective real– analytic action of* $\mathbb{Z}^3$ *on a compact (real–analytic) three–manifold is atomic.*

7.2. Regular neighborhoods and regular sets. (See [**BP**] for a more detailed discussion in the rank one case.) Consider an ergodic invariant measure μ for a local $\mathbb{Z}^k$ action F. For μ–a.e. point $x \in \Lambda$ we can construct for any $\epsilon > 0$ the ϵ–Lyapunov metric for the derivative extension. Fix a sufficiently small $\epsilon > 0$; typically it is sufficient to assume that ϵ is less than a quarter of the minimal difference between two different Lyapunov exponents of the derivative extension. There are two types of regular neighborhoods for regular points in the sense of Section 5.1. The neighborhoods of the first kind are constructed by taking "boxes" i.e. products of balls in the Lyapunov metric of a fixed sufficiently small size in the spaces $E_{\chi_i}(x)$, $i = 1, \ldots, l$ and exponentiating these boxes by the exponential map of the reference smooth Riemannian metric. The fine Lyapunov decomposition and the ϵ–Lyapunov metric for any $\epsilon > 0$ are continuous when restricted to any set where $\frac{\|u\|_{x,\epsilon}}{\|u\|}$ is bounded. In particular the sets $\Lambda_{\epsilon,\delta}$ defined by (5.1.7) are closed in this

situation and the ϵ–Lyapunov metric and the regular neighborhoods are continuous on those sets. Such a neighborhood comes provided with a standard coordinate system, i.e. a smooth embedding of the standard box B in $\mathbb{R}^m$ into M. Continuity means continuous dependence of this map from the base. point in a proper topology, typically, since we mostly deal with C^∞ maps. the topology will be the C^∞ topology. Similarly to the rank one case [**Br**] it can be shown later that these structures are in fact Hoelder continuous.

The second type of regular neighborhoods is described in [**KM**], Theorem S.3.1. These neighborhoods are obtained by "truncating" the boxes in Lyapunov metric further to make their size changing slowly with respect to the reference metric. In most situation any of the two types of regular neighborhoods may be used.

7.3. Stable and unstable manifolds. (See [**BP**].) Let us assume that μ is a hyperbolic measure. Specifying the discussion from Section 5.2 to this case we obtain the coarse Lyapunov decomposition of $T_\Lambda M$:

$$T_\Lambda M = \bigoplus_{i=1}^{s} E_{H_i}(x).$$

Hadamard-Perron Theorem for $\mathbb{Z}^k$ actions asserts that each element of this decomposition as well as distributions obtained by the upper filtration in such an element can be uniquely integrated to a family of smooth submanifolds which possesses certain regularity properties away from sets of small measure. More specifically fix a sufficiently small ϵ as before and let $\Lambda_\epsilon = \cup_{\delta>0}\Lambda_{\epsilon,\delta}$.

Let E be a distribution from an upper filtration of the coarse Lyapunov decomposition and E' be the sum of the elements of the fine Lyapunov decomposition which do not belong to E and $x \in \Lambda_\epsilon$. In the regular neighborhood $\mathcal{R}(x)$ of x of the first kind, there is a natural product structure given by the exponential of the decomposition $T_xM = E(x) \oplus E'(x)$. Then *the local integral manifold* $\mathcal{E}(x) \subset \mathcal{R}(x)$ has the form $\exp_x \operatorname{graph} \phi_x$, where $\phi_x : B(E(x)) \to E'(x)$. Here $B(E(x))$ is the box around the origin in $E(x)$ of fixed size in the Lyapunov metric used in the construction of the regular neighborhoods and ϕ_x is a smooth function (in fact, its degree of regularity is the same as for the action F), which vanishes at the origin together with its first differential. Furthermore, if $\|D_xF(n)\|_{x,\epsilon} < 1 - \epsilon$, then $F(n)(\mathcal{E}(x)) \subset \mathcal{E}(F(n)x)$ and $F(n)$ contracts $\mathcal{E}(x)$. In fact, these manifolds can be obtained by considering upper filtrations for individual regular elements of the action constructing the corresponding families of integral manifolds and taking their intersections. In the case $E = E_{H_i}$ one takes intersections of local stable manifolds for regular elements of the action.

The *global integral manifold* $\mathfrak{E}(x)$ is defined as the smallest invariant set containing the local integral manifold $\mathcal{E}(x)$, i.e.

$$\mathfrak{E}(x) = \cup_{n\in\mathbb{Z}^k} F(n)(\mathcal{E}(F^{-n}(x)).$$

Global integral manifolds are images of the Euclidean space under smooth injective immersions.

As we mentioned before in the description of the regular sets $\Lambda_{\epsilon,\delta}$ it is sufficient (and convenient) to fix a sufficiently small value of the parameter ϵ. This way we obtain a family of compact sets depending on a single parameter which we will denote by θ. This parameter roughly can be thought as the "guaranteed size" of

integral manifolds for the elements of the coarse Lyapunov decomposition and their upper filtrations.

8. Rigidity of hyperbolic measures in dimension three: an outline

The crucial feature of the argument in Section 2.2 was presence of the critical direction such that the action along that direction was isometric along the foliation W and sufficiently ergodic. If we want to apply this type of arguments in more general situation we need to understand when a similar situation may appear.

In the case of algebraic actions discussed in Part I both the guidance and the restrictions are provided by linear algebra. Now we will consider a situation which looks much more general from dynamical point of view than our basic example of an action of $\mathbb{Z}^2$ on $\mathbb{T}^3$ by hyperbolic automorphisms but whose geometric features are very similar.

8.1. Lyapunov Hoelder cocycle rigidity for TNS measures. Let us consider an action α of $\mathbb{Z}^2$ by diffeomorphisms of a compact three-dimensional manifold M. The key regularity assumption is $C^{1+\epsilon}$, for some $\epsilon > 0$, but at the moment we can assume that the action is C^∞. Consider an ergodic α-invariant measure μ such that all three Lyapunov exponents are different from zero and none of them are proportional. In other words, the three Lyapunov lines (see Section 5.2) are all different. These lines divide $\mathbb{R}^2$ into six Weyl chambers. For a μ almost every point $x \in M$ there exists a measurable α-invariant splitting of the tangent space T_xM into three one-dimensional subspaces $E_1(x)$, $E_2(x)$, and $E_3(x)$ corresponding to three Lyapunov exponents. Each of these distributions as well as their pairwise sums are uniquely integrable to α-invariant families of smooth manifolds defined almost everywhere with respect to the measure μ (See Section 7.3). While these distributions in general are not continuous, they are Hölder with respect to ϵ-Lyapunov metrics defined in Section 5.1. Fix a smooth Riemannian metric on M and let $J_i(x,a) = \log ||D\alpha(a)|_{E_i(x)}||$, $i = 1,2,3$. The function J_i are additive one cocycles, i.e. $J_i(x, a_1 + a_2) = J_i(x, a_1) + J_i(\alpha(a_1)(x), a_2)$. Naturally these cocycles are also Hölder with respect to ϵ-Lyapunov metrics. For these class of cocycles we can prove cohomological rigidity: for such a cocycle J one can find a function H, also satisfying a Hölder condition in a Lyapunov metric, and a homomorphism $\sigma : \mathbb{Z}^2 \to \mathbb{R}$ such that

$$J(x,a) = \sigma(a) + H(\alpha(a)(x)) - H(x).$$

8.2. Invariant affine structure on contracting manifolds. Let W be one of the three families of one-dimensional invariant manifolds. It turns out that there is a uniquely defined family of α-invariant smooth affine parameters on these manifolds. This is a particular case of the normal form results discussed in Section 6.2 (Theorems 6.2 and 6.3 since in the case of a single exponent there are no resonance relations. the proof is in fact more direct and simple than in the general case. The cocycle trivialization result described in the previous subsection allows to normalize the affine parameters in such a way that they are transformed by the transformation $\alpha(a)$ with the constant expansion or contraction coefficient $\exp \sigma(a)$. Now pass to the suspension of the action α which we also denote by α. This action in the direction of the corresponding Lyapunov line preserves the normalized affine parameters. From here on we can closely intimate the arguments for the

automorphisms of $\mathbb{T}^3$ from Section 2.2 exercising a certain care due to the fact that this structure is only measurable.

8.3. Conditional measures and rigidity. Consider the system of conditional measures on the leaves of W and identify a typical global leaf with $\mathbb{R}$ using a point in the support of the conditional measure as the origin and the length parameter normalized at that point. At this moment it is important to keep in mind that the normalizations at different (even typical) points of the same leaf need not agree. However by Luzin Theorem type argument they do not differ more than a fixed amount on a set of large measure. Assumption $(\mathcal{E})$ holds essentially by the same reasons as in Section 2.2. The first part of the proof of the key assertion $(\mathcal{T})$ which uses weak* compactness and Luzin theorem holds essentially *verbatim*. Thus, the conditional measure on a typical leaf of W is invariant (maybe up to a rescaling) with respect to an almost every translation. Again assuming that the conditionals are atomic implies that the every element of the action has zero entropy. In the case of continuous conditional measures one sees again that the support of it is a closed subgroup of $\mathbb{R}$, hence the whole $\mathbb{R}$. The argument showing that the rescaling cocycle is identically equal to one again holds. Thus, the conditionals are Lebesgue with respect to the smooth invariant affine parameter, hence, absolutely continuous. After that simple arguments using Ledrappier's converse [**L**] to the Pesin entropy formula [**P**] show that the measure μ itself must be absolutely continuous. Thus the conclusion is the following very general counterpart of the statement (D) from section 2.2.

THEOREM 8.1. *Let μ be an ergodic invariant measure for a smooth action of $\mathbb{Z}^2$ on a compact three-dimensional manifold M such that all three Lyapunov characteristic exponents are different from zero and none of them are proportional to each other. Then either μ is absolutely continuous or all elements of the action have zero entropy with respect to μ.*

A detailed proof of this theorem will appear in a separate paper. An important corollary is the following result which to the bet of our knowledge is the first instance in the differentiable dynamics in dimension greater than one when global data (the homotopy class in our case) force existence of an invariant geometric structure for an action of noncompact amenable group.

THEOREM 8.2. *Any $\mathbb{Z}^2$ action α on $\mathbb{T}^3$ which is homotopic to an action α_0 by hyperbolic automorphisms has an ergodic absolutely continuous invariant measure.*

PROOF. There is a surjective continuous map $\phi : \mathbb{T}^3 \to \mathbb{T}^3$ such that

$$\phi \circ \alpha = \alpha_0 \circ \phi.$$

This fact can be proven as follows. First, by a theorem of Franks ([**KH**, Theorem 2.6.1]) there is such a map homotopic to identity (call it $\phi(\mathbf{m})$) for each element $\alpha(\mathbf{m})$ for $\mathbf{m} \in \mathbb{Z}^2 \setminus \{0\}$ and it is unique. Uniqueness implies that the centralizer of any hyperbolic automorphism of a torus in the group of homeomorphisms of the torus coincides with its centralizer in the group of automorphisms; in particular, this centralizer contains no more that one element in each homotopy class. Since for any $\mathbf{m}_1, \mathbf{m}_2 \in\in \mathbb{Z}^2 \setminus \{0\}$ the map $\phi(\mathbf{m}_1)^{-1} \circ \alpha(\mathbf{m}_2) \circ \phi(\mathbf{m}_1)$ commutes with $\alpha_0(\mathbf{m}_1)$ and is homotopic to $\alpha_0(\mathbf{m}_2) = \phi(\mathbf{m}_2)^{-1} \circ \alpha(\mathbf{m}_2) \circ \phi(\mathbf{m}_2)$ we deduce that

$$\phi(\mathbf{m}_1) = \phi(\mathbf{m}_2).$$

In other words, the algebraic action α_0 is a factor of the action α, or, equivalently α is an extension of α_0.

Consider the set $\mathcal{M}$ of all Borel probability measures ν on $\mathbb{T}^3$ such that $(\phi)_*\nu = \lambda$, Lebesgue measure. The set $\mathcal{M}$ is convex, weak* compact and α invariant. Hence by Tychonoff theorem it contains an α invariant measure. Since α_0 is ergodic with respect to λ almost every ergodic component of an α-invariant measure $\nu \in \mathcal{M}$ also belongs to $\mathcal{M}$. Let μ be such an ergodic measure. We will show that it is absolutely continuous.

Since $(\phi)_*\mu = \lambda$, for any $\mathbf{m} \in \mathbb{Z}^2$, and since for each $\mathbf{m} \in \mathbb{Z}^2$ either the stable or unstable foliation for $\alpha_0(\mathbf{m})$ is one-dimensional, one has

$$h_\mu(\alpha(\mathbf{m})) \geq h_\lambda(\alpha_0(\mathbf{m})) = |\log \max |\chi_i(\mathbf{m})| \, |,$$

where $\chi_1(\mathbf{m}), \chi_2(\mathbf{m})$ and $\chi_3(\mathbf{m})$ are eigenvalues of the matrix $\rho_\alpha(\mathbf{m})$.

This implies in particular that the entropies $h_\mu(\alpha(\mathbf{m}))$ for all $\mathbf{m} \in \mathbb{Z}^2 \setminus \{0\}$ are uniformly bounded away from zero.

In order to apply Theorem 8.1 we need to show that μ is a hyperbolic measure and that all three Lyapunov lines are different. Notice that this is of course true for the Lebesgue measure with respect to α_0; after all, this was the starting point of the argument in Section 2.2. If there are no more than two Lyapunov lines for α then along at least one of these lines at least two of the three Lyapunov exponents vanish. By Ruelle inequality [**KM**] this means that the entropy of the elements of the suspension along that line vanishes. Since there are elements of $\mathbb{Z}^2$ either on the line (if the latter is rational) or arbitrary close to it (if it is irrational), there are nonzero elements of α with arbitrary small entropy, a contradiction. □

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