CHAPTER 11

Spectral Properties and Combinatorial Constructions in Ergodic Theory

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This survey primarily deals with certain aspects of ergodic theory, i.e. the study of groups of measure preserving transformations of a probability (Lebesgue) space up to a metric isomorphism [8, Section 3.4a]. General introduction to ergodic theory is presented in [8, Section 3]. Most of that section may serve as a preview and background to the present work. Accordingly we will often refer to definitions, results and examples discussed there. For the sake of convenience we reproduce some of the basic material here as need arises.

Here we will deal exclusively with actions of Abelian groups; for a general introduction to ergodic theory of locally compact groups as well as in-depth discussion of phenomena peculiar to certain classes of non-Abelian groups see [4]. Furthermore, we mostly concentrate on the classical case of cyclic systems, i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$. Differences between those cases and the higher-rank situations (basically $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geqslant 2$) appear already at the measurable level but are particularly pronounced when one takes into account additional structures (e.g., smoothness).

Expository work on the topics directly related to those of the present survey includes the books by Cornfeld, Fomin and Sinai [29], Parry [124], Nadkarni [114], Queffelec [128], and the first author [78] and surveys by Lemańczyk [104] and Goodson [64]. Our bibliography is far from comprehensive. Its primary aim is to provide convenient references where proofs of results stated or outlined in the text could be found and the topics we mention are developed to a greater depth. So we do not make much distinction between original and expository sources. Accordingly our references omit original sources in many instances. We make comments about historical development of the methods and ideas described only occasionally. These deficiencies may be partially redeemed by looking into expository sources mentioned above. We recommend Nadkarni's book and Goodson's survey in particular for many references which are not included to our bibliography. Goodson's article also contains many valuable historical remarks.

1. Spectral theory for Abelian groups of unitary operators

1.1. *Preliminaries*

1.1.1. *Spectral vs. metric isomorphism.* Any measure preserving action Φ of a group G on a measure space (X, μ) generates a unitary representation of G in the Hilbert space $L^2(X, \mu)$ by $U_g : \varphi \mapsto \varphi \circ \Phi^{g^{-1}}$. For an action of $\mathbb{Z}$ generated by $T : X \to X$ the notation U_T for the operator U_1 is commonly used; often this operator is called Koopman operator since this connection was first observed in [95]. If two actions are isomorphic then the corresponding unitary representations in L^2 are unitarily equivalent, hence any invariant of unitary equivalence of such operators defines an invariant of isomorphism. Such invariants are said to be *spectral invariants* or *spectral properties*. Actions for which the corresponding unitary representations are unitarily equivalent are usually called *spectrally isomorphic*. We will use terms "unitarily equivalent" and "unitarily isomorphic" interchangeably.

Let us quickly describe the difference between the spectral and metric isomorphism for groups of unitary operators generated by measure preserving actions. In addition to the structure of Hilbert space which is preserved by any unitary operator, the space $L^2(X, \mu)$